TRANSFORMATIONS

A TYPES OF TRANSFORMATIONS

Transformations are rules that take every point of a figure to another point in the plane. They can move (translate), flip (reflect), turn (rotate), or resize (enlarge/reduce) a shape.

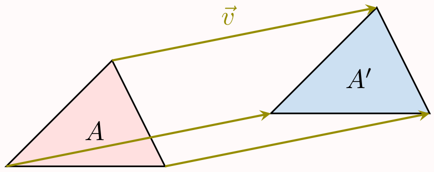
Definition Object and Image

When a transformation is applied to a shape, the original shape is called the **object**.
The resulting shape after the transformation is called the **image**.
Often, if a point is called A in the object, its image is written A' ("A prime").

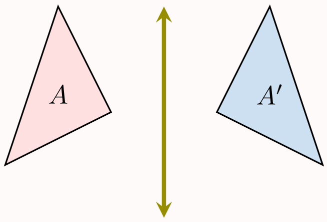
Definition Types of Transformations

There are several types of transformations, including:

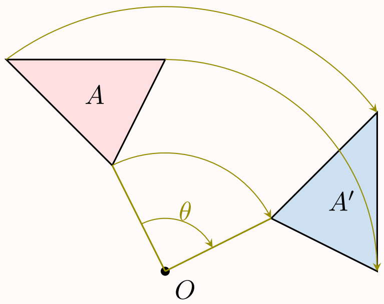
- **Translation:** Slides every point of a shape the same distance in the same direction.
  It does not change the shape or size of the figure and does not change the orientation (it is a rigid motion).



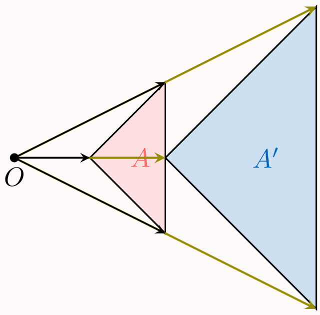
- **Reflection:** Flips a shape across a line (like a mirror), creating a mirror image.
  This line is called the *axis (line) of reflection.*



- **Rotation:** Turns a shape around a fixed point (the *centre of rotation*) by a certain angle.
  A positive angle is usually taken to mean a counterclockwise rotation. Distances and angles are preserved.



- **Homothety (enlargement/reduction):** Resizes a shape from a centre point by a constant scale factor, so that the image is *similar* to the original (angles are preserved, lengths are multiplied by the same factor).

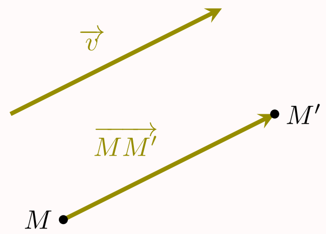


B TRANSLATION

A **translation** moves a figure from one place to another. Every point on the figure moves the same distance in the same direction.

Definition Translation

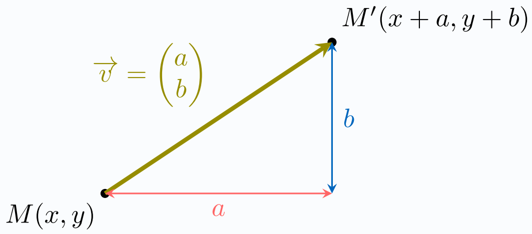
A translation by the vector $\overrightarrow{v}$ maps a point M to its image M' such that $\overrightarrow{MM'} = \overrightarrow{v}$.
All points of the plane are shifted by the same vector $\overrightarrow{v}$.



Proposition Coordinates of the Image Point

In a coordinate system, if the point M has coordinates (x, y) and the translation vector is $\overrightarrow{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, then the image point M' has coordinates

$$M'(x + a,\ y + b).$$



Proof

Let M' be the image point with coordinates (x', y'). Then

$$\overrightarrow{MM'} = \overrightarrow{v}$$
$$\begin{pmatrix} x' - x \\ y' - y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$
$$x' - x = a \quad \text{and} \quad y' - y = b$$
$$x' = x + a \quad \text{and} \quad y' = y + b.$$

Therefore, $M'(x + a, y + b)$.

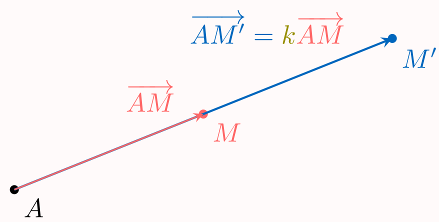
C HOMOTHETY

Definition Homothety

A **homothety** with center A and scale factor k maps a point M to a point M' on the line AM such that

$$\overrightarrow{AM'} = k\,\overrightarrow{AM}.$$

If $|k| > 1$, the figure is enlarged; if $0 < |k| < 1$, the figure is reduced.



Proposition Coordinates of the Image Point

In a coordinate system, if the center A has coordinates (a, b), the point M has coordinates (x, y), and the scale factor is k, then the image point M' has coordinates

$$M'\big(a + k(x-a),\ b + k(y-b)\big).$$

Proof

Let M' be the image point with coordinates (x', y'). Then

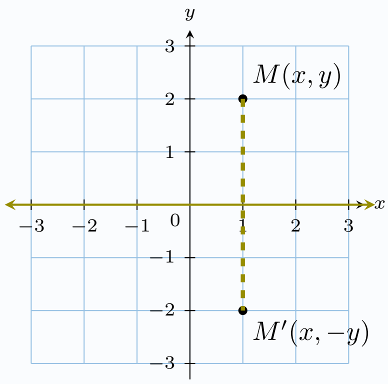
$$\begin{aligned} \overrightarrow{AM'} &= k\overrightarrow{AM} \\ \begin{pmatrix} x'-a \\ y'-b \end{pmatrix} &= k\begin{pmatrix} x-a \\ y-b \end{pmatrix} \\ x'-a = k(x-a) \quad &\text{and} \quad y'-b = k(y-b) \\ x' = a + k(x-a) \quad &\text{and} \quad y' = b + k(y-b). \end{aligned}$$

So M' has coordinates $\big(a + k(x-a), b + k(y-b)\big)$.

D SPECIFIC REFLECTIONS

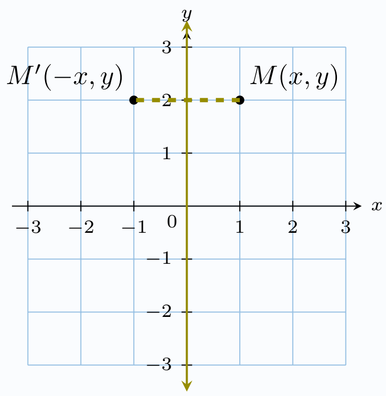
Proposition Reflection over the x-axis

The image of the point $M(x, y)$ under the reflection over the x-axis is $M'(x, -y)$.
This reflection keeps the x-coordinate and changes the sign of the y-coordinate.



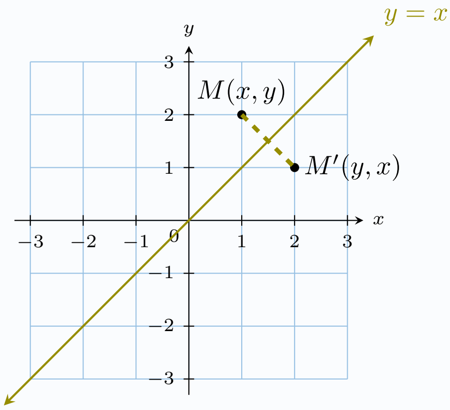
Proposition Reflection over the y-axis

The image of the point $M(x, y)$ under the reflection over the y-axis is $M'(-x, y)$.
This reflection keeps the y-coordinate and changes the sign of the x-coordinate.



Proposition Reflection over the line $y = x$

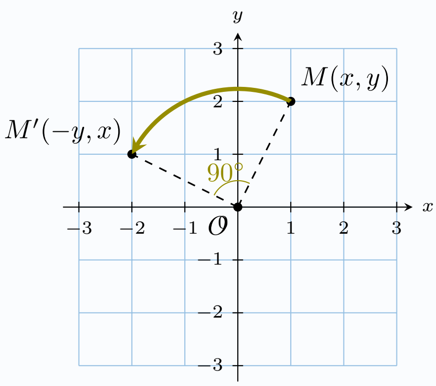
The image of the point $M(x, y)$ under the reflection $M_{y=x}$ over the line $y = x$ is $M'(y, x)$.
The coordinates are swapped.



E SPECIFIC ROTATIONS

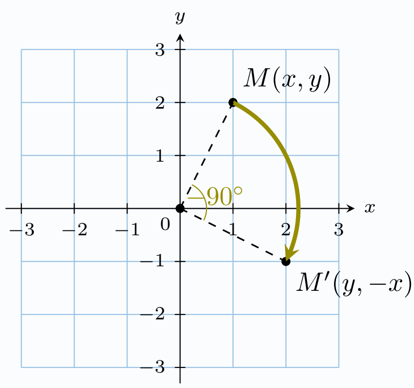
Proposition Rotation of 90°

The image of the point $M(x, y)$ under the rotation of 90° (counterclockwise) around the origin is $M'(-y, x)$.
In coordinates, a 90° anticlockwise rotation sends (x, y) to $(-y, x)$.



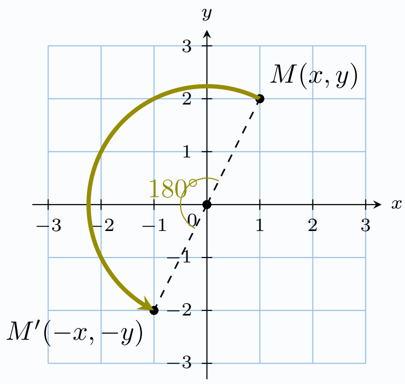
Proposition Rotation of -90°

The image of the point $M(x, y)$ under the rotation of $-90°$ (clockwise) around the origin is $M'(y, -x)$.
In coordinates, a 90° clockwise rotation sends (x, y) to $(y, -x)$.



Proposition Rotation of 180°

The image of the point $M(x, y)$ under the rotation of 180° around the origin is $M'(-x, -y)$.
In coordinates, a half-turn about the origin sends (x, y) to $(-x, -y)$.



y
x
M(x, y)
M′(−x, −y)
180°
0
−3
−2
−1
1
2
3