FUNCTIONS

A FUNDAMENTAL CONCEPTS OF FUNCTIONS

A.1 WHAT IS A FUNCTION?

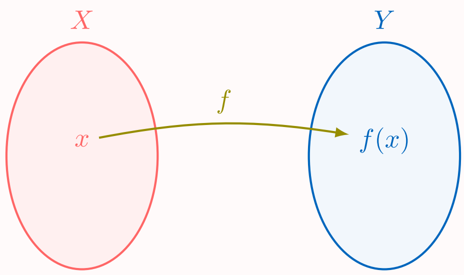
Definition Function

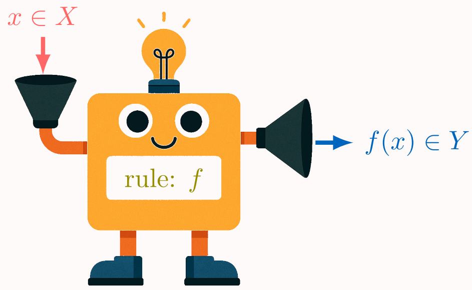
A **function** $f : X \to Y$ is a rule that assigns to each element x in a set X exactly one element $f(x)$ in a set Y.

- The set X of all possible inputs is called the **domain** of f.
- The set Y is called the **codomain** of f.

We write $f : x \mapsto f(x)$ to indicate the rule that maps an element x to its corresponding image $f(x)$.

$$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto f(x) \end{aligned}$$





- f is the name of the function.
- x is the input variable, an element from the domain.
- $f(x)$ is the output value in the codomain when the input is x. It is read as "f of x".
- $f(x)$ is the **image** of x under f.
- x is a **preimage** of $y = f(x)$.

Ex: Let the function $\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto 2x - 1 \end{aligned}$. Find the image of 5 under f.

Answer: To find $f(5)$, we substitute the input value $x = 5$ into the function's rule:

$$\begin{aligned} f(5) &= 2(5) - 1 \\ &= 10 - 1 \\ &= \mathbf{9} \end{aligned}$$

Method Finding Inputs from Outputs Algebraically

To find the preimage(s) of a value y for a function $f(x)$:

- Set the function's formula equal to the output value: $\boldsymbol{f(x) = y}$.
- Solve the resulting equation for x.

Ex: Let $f(x) = 3x + 12$. Find x such that $f(x) = 0$.

Answer: We need to find the value of x such that $f(x) = 0$. We set up the equation and solve:

$$\begin{aligned} f(x) &= 0 \\ 3x + 12 &= 0 \\ 3x &= -12 && \text{(subtract 12 from both sides)} \\ x &= \frac{-12}{3} && \text{(divide both sides by 3)} \\ x &= -4 \end{aligned}$$

The preimage of 0 is $\boldsymbol{x = -4}$.
Check: $f(-4) = 3(-4) + 12 = -12 + 12 = 0$. The answer is correct.

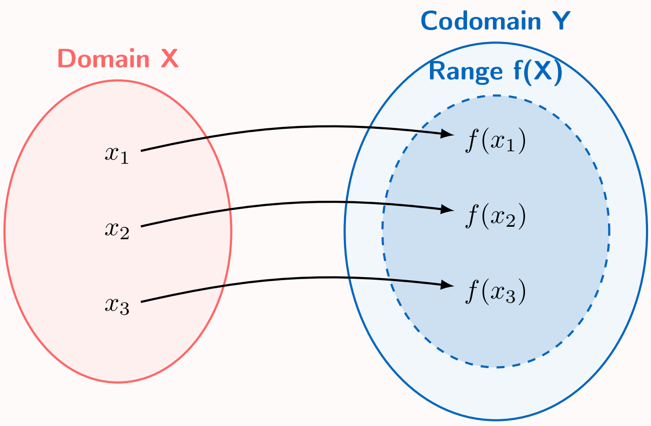
A.2 NATURAL DOMAIN AND RANGE

Definition Range

The **range** of a function $f : X \to Y$ is the set of all actual output values produced by the function. It is the set of all images of the elements in the domain.

$$\text{Range} = f(X) = \{f(x) : x \in X\}$$

The range is a **subset** of the codomain ($f(X) \subseteq Y$). While the codomain is the set of **potential** outputs, the range is the set of **actual** outputs.



Ex: For the function $\begin{array}{rccl} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & x^2 \end{array}$:

- The **domain** is $\mathbb{R}$ (all real numbers).
- The **codomain** is also $\mathbb{R}$.
- However, since $x^2 \geq 0$, the **range** is $[0, \infty)$, which is a subset of the codomain.

Definition Natural Domain

When a function is given by a formula and the domain is not explicitly specified, the **natural domain** is the largest set of real numbers for which the formula yields a defined real number.

Note When a function's rule is given without explicitly specifying the domain and codomain, we assume the **natural domain** as the domain and the set of all real numbers, $\mathbb{R}$, as the codomain.
For example, when we write $f : x \mapsto \sqrt{x}$, we implicitly assume the domain is the set of non-negative real numbers, $[0, \infty)$, and the codomain is $\mathbb{R}$. Thus, it is shorthand for $\begin{array}{rccl} f: & [0,\infty) & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \sqrt{x} \end{array}$.
Sometimes, we simply refer to "the function $\sqrt{x}$". This is also a shorthand for the function defined on its natural domain.

Method Finding the Natural Domain

To find the natural domain of a function, we assume the domain is all real numbers ($\mathbb{R}$) and then exclude any values of x that would lead to an undefined mathematical operation. At this level, we look for restrictions caused by:

1. **Rational Functions:** The denominator of a fraction cannot be zero. We solve 'denominator $= 0$' to find values to exclude.

2. **Even Roots:** The expression inside an even root (like a square root, $\sqrt{\cdot}$, or fourth root, $\sqrt[4]{\cdot}$) must be non-negative (≥ 0).
3. **Logarithms:** The argument of a logarithm must be strictly positive (> 0). (Note: This will be covecolordef in more detail in the logarithms chapter).

Ex: Find the natural domain of the function $f : x \mapsto \frac{1}{x-2}$.

Answer: The function involves division. Division is undefined when the denominator is zero. Therefore, we must exclude any value of x that makes the denominator $x - 2$ equal to zero.
Set the denominator to zero and solve for x:

$$x - 2 = 0 \Leftrightarrow x = 2$$

The natural domain is the set of all real numbers except 2. Using set notation, we write:

$$\text{Domain} = \{x \in \mathbb{R} \mid x \neq 2\} \quad \text{or} \quad \mathbb{R} \setminus \{2\}$$

In interval notation, this is $(-\infty, 2) \cup (2, \infty)$.

A.3 TABLES OF VALUES

Definition Table of Values

A **table of values** is a table that organizes the relationship between the input values (x) and their corresponding output values ($f(x)$) for a function.

Ex: Complete the table of values for the function $f : x \mapsto x^2$.

x	-2	-1	0	1	2
$f(x)$					

Answer: We substitute each value of x into the function $f(x) = x^2$:

- $f(-2) = (-2)^2$
  $= 4$
- $f(-1) = (-1)^2$
  $= 1$
- $f(0) = (0)^2$
  $= 0$
- $f(1) = (1)^2$
  $= 1$
- $f(2) = (2)^2$
  $= 4$

The completed table is:

x	-2	-1	0	1	2
$f(x)$	4	1	0	1	4

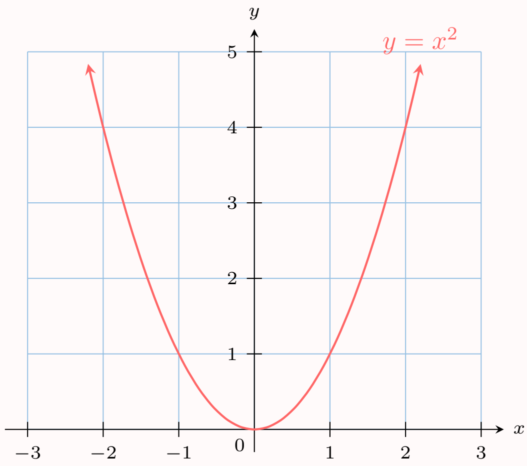
A.4 GRAPHS

While a table of values is useful for listing some input–output pairs of a function, a **graph** is a powerful tool for visualizing how the output changes when the input changes. A graph gives us a picture of the function's behavior.

Definition Graph of a Function

The **graph** of a function f is the set of all points with coordinates $(x, f(x))$ plotted on a coordinate plane. The input, x, is plotted on the horizontal axis (the x-axis), and the output, $f(x)$, is plotted on the vertical axis (the y-axis). When we connect these points, we form the curve of the function.

$$\text{Graph of } f = \{(x, f(x)) : x \in X\}$$



Method Plotting a Graph from a Table

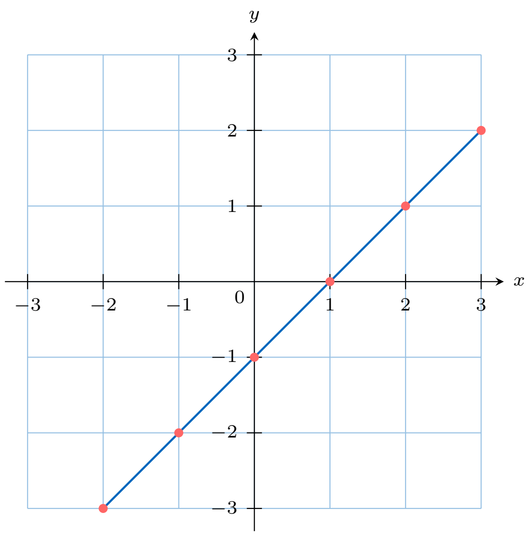
To plot the graph of a function from its table of values:

1. Draw a coordinate plane with a suitable scale on each axis and label the axes.
2. For each pair $(x, f(x))$ in the table, plot the corresponding point on the coordinate plane.
3. If the function is defined for all x in the interval shown, connect the points with a straight line or a smooth curve.

Ex: Plot the graph of the function $f(x) = x - 1$ using its table of values.

x	-2	-1	0	1	2	3
$f(x)$	-3	-2	-1	0	1	2

Answer: We plot the points $(-2, -3)$, $(-1, -2)$, $(0, -1)$, $(1, 0)$, $(2, 1)$, and $(3, 2)$ from the table. These points lie on the same straight line, so we connect them to draw the graph of $f(x) = x - 1$.



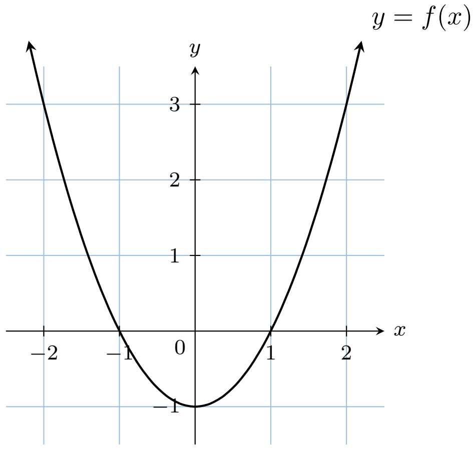
Method Finding the Value of $f(x)$ from a Graph

To find the output $f(x)$ for a given input x using a graph:

1. **Locate the input value** on the horizontal x-axis.
2. **Move vertically** from that point until you reach the curve of the function.

3. **Move horizontally** from the intersection point to the vertical y-axis and read the corresponding value. This y-value is the output $f(x)$.

Ex: Using the graph of the function f below, find the value of $f(2)$.

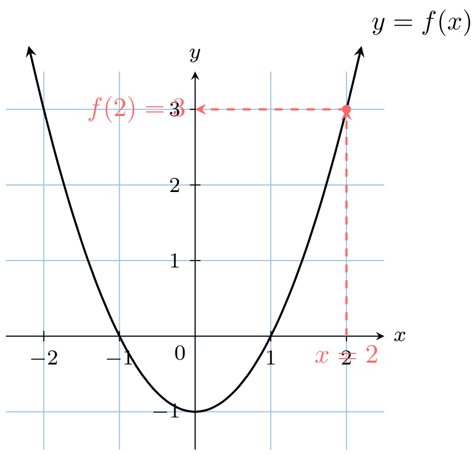


Answer: We follow the graphical method:

1. Start at $x = 2$ on the horizontal axis.

2. Move up to meet the curve.

3. Move horizontally to the vertical axis and read the value, which is 3.



Therefore, $\boldsymbol{f(2) = 3}$.

We have learned how to take an input (x) and find its output $(f(x))$. Now, we will learn how to work backwards: if we know the output, can we find the input(s) that produced it? This process is called finding the **preimage(s)** (or input(s)) of a given value.

Method **Finding Inputs from Outputs on a Graph**

To find the preimage(s) of a value y (i.e., find all x such that $f(x) = y$):

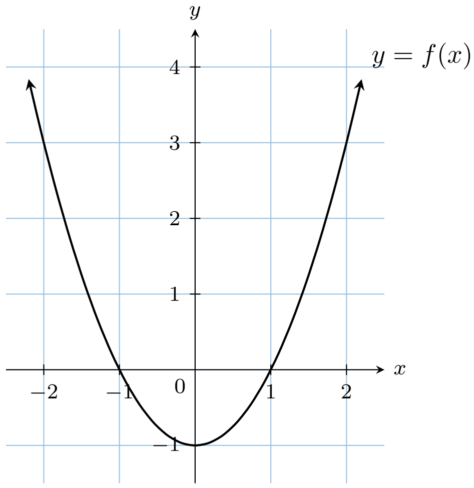
1. **Locate the output value** y on the vertical y-axis.

2. **Draw a horizontal line** from this value across the graph.

3. **Find the intersection point(s)** where the horizontal line crosses the function's curve.

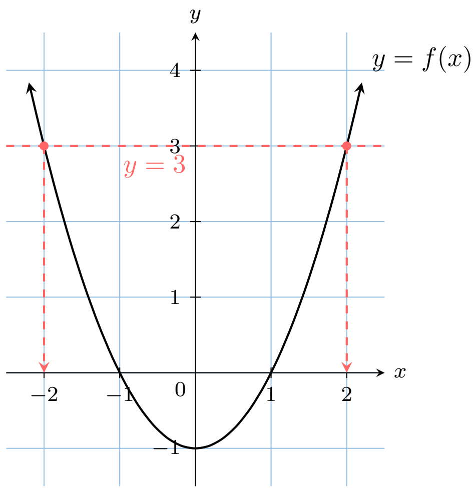
4. **Move vertically to the x-axis** from each intersection point to read the corresponding input value(s). These are the preimages.

Ex: Using the graph of the function f below, find x such that $f(x) = 3$.



Answer: We apply the graphical method:

1. We locate $y = 3$ on the vertical axis.
2. We draw a horizontal line at $y = 3$.
3. The line intersects the curve at two points.
4. We move vertically from these points to the x-axis to read the values, which are -2 and 2.



The preimages of 3 are $\mathbf{-2}$ and $\mathbf{2}$.

Finding a preimage graphically is useful for visualization, but for an exact answer, we can use algebra.

A.5 BIJECTIVE FUNCTIONS

For a function to be perfectly reversible (i.e., to have an inverse), it must create a perfect pairing between the elements of its domain and its codomain. This leads to the concept of a bijection, which is a function that is both injective and surjective.

Definition Injective, Surjective, and Bijective Functions

Let $f : X \to Y$ be a function.

- f is **injective (one-to-one)** if different inputs produce different outputs. For any $x_1, x_2 \in X$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.
- f is **surjective (onto)** if its range is equal to its codomain ($f(X) = Y$). This means every element in the codomain Y is the image of at least one element from the domain X.
- f is **bijective** if it is both **injective and surjective**. This means every element in the codomain is the image of **exactly one** element from the domain.

Ex: Consider two versions of the squaring function:

1. $$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2 \end{aligned}$$

2. $g : [0, \infty) \longrightarrow [0, \infty)$
   $x \longmapsto x^2$

- The function f is **not bijective**. It is not injective because $f(-2) = f(2)$. It is not surjective because its range, $[0, \infty)$, is not equal to its codomain, $\mathbb{R}$.

- The function g **is bijective**. It is injective because its domain is restricted to non-negative numbers. It is surjective because its range, $[0, \infty)$, is equal to its specified codomain, $[0, \infty)$.

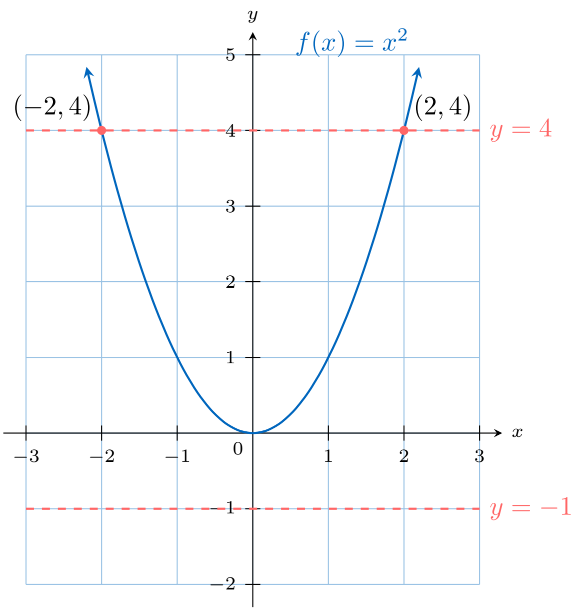
Method The Horizontal Line Test for Bijectivity

To determine visually if a function $f : X \to Y$ is bijective:

1. Draw the graph of the function $f(x)$.
2. Imagine drawing horizontal lines across the graph for **every** possible y-value in the **codomain Y**.
3. **Conclusion:**
   - The function is **bijective** if and only if **every** horizontal line (for all $y \in Y$) intersects the graph **exactly once**.
     - Intersecting **at most** once shows it is injective.
     - Intersecting **at least** once shows it is surjective.

Ex: Use the horizontal line test to determine if the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x^2$ is a bijective function.

Answer: First, we graph the function $f(x) = x^2$. The specified codomain is $\mathbb{R}$.



We test for both conditions:

- **Injectivity:** The horizontal line $y = 4$ intersects the graph at two distinct points, $(-2, 4)$ and $(2, 4)$. Since a horizontal line intersects the graph more than once, the function is **not injective**.

- **Surjectivity:** The horizontal line $y = -1$ is in the codomain ($\mathbb{R}$) but does not intersect the graph at all. This means that -1 has no preimage. Since there is an element in the codomain that is not in the range, the function is **not surjective**.

Since the function is neither injective nor surjective, it is **not bijective**.

B OPERATIONS ON FUNCTIONS

B.1 ALGEBRA OF FUNCTIONS

Definition Operations on Functions

Given two functions f and g, we can define new functions by performing arithmetic operations on their outputs, for each x where both are defined:

- **Sum:** $(f+g)(x) = f(x) + g(x)$
- **Difference:** $(f-g)(x) = f(x) - g(x)$
- **Product:** $(fg)(x) = f(x) \times g(x)$
- **Quotient:** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided that $g(x) \neq 0$.

For the sum, difference and product, the domain of the new function is the intersection of the domains of f and g (all x for which *both* $f(x)$ and $g(x)$ are defined).
For the quotient, the domain is the intersection of the domains of f and g, *excluding* any x such that $g(x) = 0$.

Ex: Let $f(x) = 2x + 1$ and $g(x) = x^4 - 1$. Find $(f+g)(x)$.

Answer:

$$\begin{aligned}(f+g)(x) &= f(x) + g(x)\\ &= (2x+1) + (x^4 - 1)\\ &= x^4 + 2x.\end{aligned}$$

Since both f and g are polynomials, $(f+g)(x)$ is defined for all real numbers x.

B.2 COMPOSITION OF FUNCTIONS

Definition Composition of Functions

Given two functions f and g, the **composite function**, denoted $\boldsymbol{f \circ g}$ (read "f composed with g"), is defined by:

$$\boldsymbol{(f \circ g)(x) = f(g(x))}$$

for every x that belongs to the domain of g and for which $g(x)$ belongs to the domain of f.
We first apply the function g to x, and then apply the function f to the result $g(x)$.

Ex: Let $f(x) = x^2$ and $g(x) = 2x + 1$.

1. Find $(f \circ g)(x)$.
2. Find $(g \circ f)(x)$.
3. Is composition commutative? (i.e., is $(f \circ g)(x) = (g \circ f)(x)$?)

Answer:

1. $$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(2x+1)\\ &= (2x+1)^2\\ &= 4x^2 + 4x + 1.\end{aligned}$$
2. $$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g(x^2)\\ &= 2(x^2) + 1\\ &= 2x^2 + 1.\end{aligned}$$
3. No. Since $4x^2 + 4x + 1 \neq 2x^2 + 1$, the two composite functions are different, so function composition is **not** commutative in general.

B.3 INVERSE FUNCTIONS

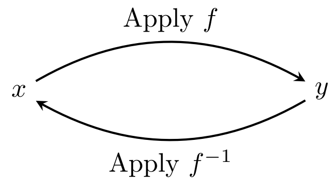
In arithmetic, we are familiar with **inverse operations**. For example, subtraction is the inverse of addition because it "undoes" the addition. If you start with 5, add 3 to get 8, and then subtract 3, you return to 5:

$$(5+3)-3=5.$$

Similarly, division is the inverse of multiplication:

$$(5\times 3)\div 3=5.$$

The concept of an **inverse function** follows the same idea. An inverse function, denoted $\boldsymbol{f^{-1}}$, is a function that "undoes" or reverses the action of another function, f.
If a function f takes an input x to an output y, the inverse function f^{-1} takes that output y back to the original input x. This creates a perfect loop:

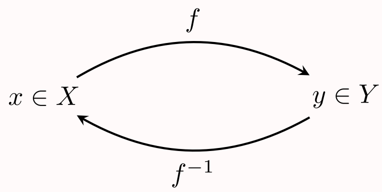


However, *not every function has* an inverse function. For an inverse to exist, each output y must come from *exactly one* input x (the function must never take the same value twice on its domain).

Definition Inverse Function

Let $f: X \to Y$ be a bijective function.
The **inverse function**, denoted f^{-1}, is the function $\boldsymbol{f^{-1}: Y \to X}$ that reverses the action of f.
This means that if f maps an input x to an output y, then f^{-1} maps that output y back to the original input x.
This relationship is defined by:

$$f(x)=y \iff f^{-1}(y)=x$$



The composition of a function and its inverse results in the identity function (which returns the original input):

- $\boldsymbol{(f^{-1}\circ f)(x)=x}$ for all x in the domain of f (X).
- $\boldsymbol{(f\circ f^{-1})(y)=y}$ for all y in the domain of f^{-1} (Y).

Method Finding the Inverse Function

To find the inverse of a function f (when it exists):

1. Set $y=f(x)$.
2. Solve the equation for x in terms of y. This gives an expression of the form $x=f^{-1}(y)$.
3. Swap the variables x and y to write the inverse in terms of x. The result is $y=f^{-1}(x)$.

This procedure defines an inverse function only if each output corresponds to exactly one input (i.e., if f is invertible on its domain).

Ex: Find the inverse of the function $f: [0,\infty) \longrightarrow [0,\infty)$.
$$x \longmapsto \sqrt{x}$$

Answer: The function is $f(x)=\sqrt{x}$ with Domain: $[0,\infty)$ and Range: $[0,\infty)$.

1. Set $y=\sqrt{x}$.
2. Solve for x: Since the domain of f is non-negative, $x\geq 0$, and the range is non-negative, $y\geq 0$. We can square both sides:
$$y^2=x \Leftrightarrow x=y^2$$

3. Swap variables to get the rule: $y = x^2$.

4. The domain of f^{-1} is the range of f, which is $[0, \infty)$. The range of f^{-1} is the domain of f, which is $[0, \infty)$.

So, the inverse function is $\boldsymbol{f^{-1} : [0, \infty) \longrightarrow [0, \infty)}$.
$$\boldsymbol{x \longmapsto x^2}$$

Proposition Symmetry of Inverse Functions

The graph of a function f and its inverse f^{-1} are reflections of each other across the line $\boldsymbol{y = x}$.