FUNCTIONS

A FUNDAMENTAL CONCEPTS OF FUNCTIONS

A.1 WHAT IS A FUNCTION?

A.1.1 WRITING FUNCTIONS: LEVEL 1

Ex 1: Consider the following calculation program:

1. Choose a number.
2. Subtract 5 from the chosen number.

Let x be the number chosen initially. Determine the function f that corresponds to the result obtained with this program.

$$f(x) = \boxed{x-5}$$

Answer: Given the following program:

1. Choose a number: x.
2. Subtract 5 from the chosen number: $x-5$.

Thus, the function is:

$$f(x) = x-5$$

Ex 2: Consider the following calculation program:

1. Choose a number.
2. Multiply the chosen number by three.

Let x be the number chosen initially. Determine the function f that corresponds to the result obtained with this program.

$$f(x) = \boxed{3x}$$

Answer: Given the following program:

1. Choose a number: x.
2. Multiply the chosen number by three: $3x$.

Thus, the function is:

$$f(x) = 3x$$

Ex 3: Consider the following calculation program:

1. Choose a number.
2. Multiply the chosen number by five.
3. Subtract 2 from the result obtained.

Let x be the number chosen initially. Determine the function f that corresponds to the result obtained with this program.

$$f(x) = \boxed{5x-2}$$

Answer: Given the following program:

1. Choose a number: x.
2. Multiply the chosen number by five: $5x$.
3. Subtract 2 from the result obtained: $5x-2$.

Thus, the function is:

$$f(x) = 5x-2$$

Ex 4: Consider the following calculation program:

1. Choose a number.
2. Multiply the chosen number by -2.
3. Add 5 to the result obtained.

Let x be the number chosen initially. Determine the function f that corresponds to the result obtained with this program.

$$f(x) = \boxed{-2x+5}$$

Answer: Given the following program:

1. Choose a number: x.
2. Multiply the chosen number by -2: $-2x$.
3. Add 5 to the result obtained: $-2x+5$.

Thus, the function is:

$$f(x) = -2x+5$$

A.1.2 WRITING FUNCTIONS: LEVEL 2

Ex 5: Consider the following calculation program:

1. Choose a number.
2. Multiply the chosen number by itself.
3. Subtract 1 from the result obtained.

Let x be the number chosen initially. Determine the function f that corresponds to the result obtained with this program.

$$f(x) = \boxed{x^2-1}$$

Answer: Given the following program:

1. Choose a number: x.
2. Multiply the chosen number by itself: x^2.
3. Subtract 1 from the result obtained: x^2-1.

Thus, the function is:

$$f(x) = x^2-1$$

Ex 6: Consider the following calculation program:

1. Choose a number.
2. Square the chosen number.
3. Multiply the result by 2.

Let x be the number chosen initially. Determine the function f that corresponds to the result obtained with this program.

$$f(x) = \boxed{2x^2}$$

Answer: Given the following program:

1. Choose a number: x.
2. Square the chosen number: x^2.
3. Multiply the result by 2: $2x^2$.

Thus, the function is:

$$f(x) = 2x^2$$

Ex 7: Consider the following calculation program:

1. Choose a number.
2. Subtract 1 from the chosen number.
3. Multiply the result by the original number chosen.

Let x be the number chosen initially. Determine the function f that corresponds to the result obtained with this program.

$$f(x) = \boxed{(x-1)x}$$

Answer: Given the following program:

1. Choose a number: x.
2. Subtract 1 from the chosen number: $x - 1$.
3. Multiply the result by the original number: $(x-1)x$.

Thus, the function is:

$$f(x) = (x-1)x$$

A.1.3 CALCULATING $f(x)$

Ex 8: For $f(x) = x + 3$,

$$f(4) = \boxed{7}$$

Answer:

$$\begin{aligned} f(4) &= (4) + 3 \quad \text{(substituting } x \text{ with } (4)) \\ &= 4 + 3 \\ &= 7 \end{aligned}$$

Ex 9: For $f(x) = 2x - 1$,

$$f(5) = \boxed{9}$$

Answer:

$$\begin{aligned} f(5) &= 2 \times (5) - 1 \quad \text{(substituting } x \text{ with } (5)) \\ &= 10 - 1 \\ &= 9 \end{aligned}$$

Ex 10: For $f(x) = 3x + 2$,

$$f(2) = \boxed{8}$$

Answer:

$$\begin{aligned} f(2) &= 3 \times (2) + 2 \quad \text{(substituting } x \text{ with } (2)) \\ &= 6 + 2 \\ &= 8 \end{aligned}$$

Ex 11: For $f(x) = x^2 - 1$,

$$f(3) = \boxed{8}$$

Answer:

$$\begin{aligned} f(3) &= (3)^2 - 1 \quad \text{(substituting } x \text{ with } (3)) \\ &= 9 - 1 \\ &= 8 \end{aligned}$$

Ex 12: For $f(x) = 5x - 3$,

$$f(1) = \boxed{2}$$

Answer:

$$\begin{aligned} f(1) &= 5 \times (1) - 3 \quad \text{(substituting } x \text{ with } (1)) \\ &= 5 - 3 \\ &= 2 \end{aligned}$$

Ex 13: For $f(x) = \frac{x}{2} + 4$,

$$f(6) = \boxed{7}$$

Answer:

$$\begin{aligned} f(6) &= \frac{(6)}{2} + 4 \quad \text{(substituting } x \text{ with } (6)) \\ &= 3 + 4 \\ &= 7 \end{aligned}$$

Ex 14: For $f(x) = x - 5$,

$$f(10) = \boxed{5}$$

Answer:

$$\begin{aligned} f(10) &= (10) - 5 \quad \text{(substituting } x \text{ with } (10)) \\ &= 10 - 5 \\ &= 5 \end{aligned}$$

Ex 15: For $f(x) = 2x - 5$,

$$f(-2) = \boxed{-9}$$

Answer:

$$\begin{aligned} f(-2) &= 2 \times (-2) - 5 \quad \text{(substituting } x \text{ with } (-2)) \\ &= -4 - 5 \\ &= -9 \end{aligned}$$

Ex 16: For $f(x) = -x + 4$,

$$f(-3) = \boxed{7}$$

Answer:

$$\begin{aligned} f(-3) &= -(-3) + 4 \quad \text{(substituting } x \text{ with } (-3)) \\ &= 3 + 4 \\ &= 7 \end{aligned}$$

Ex 17: For $f(x) = 3x - 7$,

$$f(-1) = \boxed{-10}$$

Answer:

$$\begin{aligned} f(-1) &= 3 \times (-1) - 7 \quad \text{(substituting } x \text{ with } (-1)) \\ &= -3 - 7 \\ &= -10 \end{aligned}$$

Ex 18: For $f(x) = x^2 - 2x$,

$$f(-2) = \boxed{8}$$

Answer:

$$\begin{aligned} f(-2) &= (-2)^2 - 2 \times (-2) \quad \text{(substituting } x \text{ with } (-2)) \\ &= 4 + 4 \\ &= 8 \end{aligned}$$

Ex 19: For $f(x) = 2x + 3$,

$$f(-3) = \boxed{-3}$$

Answer:

$$\begin{aligned} f(-3) &= 2 \times (-3) + 3 \quad \text{(substituting } x \text{ with } (-3)) \\ &= -6 + 3 \\ &= -3 \end{aligned}$$

Ex 20: For $f(x) = \frac{x}{2} - 4$,

$$f(8) = \boxed{0}$$

Answer:

$$\begin{aligned} f(8) &= \frac{(8)}{2} - 4 \quad \text{(substituting } x \text{ with } (8)) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

Ex 21: For $f(x) = \frac{3x-5}{2}$,

$$f(-1) = \boxed{-4}$$

Answer:

$$\begin{aligned} f(-1) &= \frac{3 \times (-1) - 5}{2} \quad \text{(substituting } x \text{ with } (-1)) \\ &= \frac{-3 - 5}{2} \\ &= \frac{-8}{2} \\ &= -4 \end{aligned}$$

Ex 22: For $f(x) = \frac{x-6}{2} - 3$,

$$f(10) = \boxed{-1}$$

Answer:

$$\begin{aligned} f(10) &= \frac{(10) - 6}{2} - 3 \quad \text{(substituting } x \text{ with } (10)) \\ &= \frac{4}{2} - 3 \\ &= 2 - 3 \\ &= -1 \end{aligned}$$

A.1.4 CALCULATING $f(x)$

Ex 23: For $f : x \mapsto x + 3$,

$$f(4) = \boxed{7}$$

Answer:

$$\begin{aligned} f(4) &= (4) + 3 \quad \text{(substituting } x \text{ with } (4)) \\ &= 4 + 3 \\ &= 7 \end{aligned}$$

Ex 24: For $f : x \mapsto x^2 - 1$,

$$f(2) = \boxed{3}$$

Answer:

$$\begin{aligned} f(2) &= (2)^2 - 1 \quad \text{(substituting } x \text{ with } (2)) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

Ex 25: For $f : x \mapsto (x-1)(x-2)$,

$$f(0) = \boxed{2}$$

Answer:

$$\begin{aligned} f(0) &= (0-1)(0-2) \quad \text{(substituting } x \text{ with } (0)) \\ &= (-1) \times (-2) \\ &= 2 \end{aligned}$$

Ex 26: For $f : x \mapsto x^3$,

$$f(-1) = \boxed{-1}$$

Answer:

$$\begin{aligned} f(-1) &= (-1)^3 \quad \text{(substituting } x \text{ with } (-1)) \\ &= -1 \end{aligned}$$

A.1.5 EVALUATING FUNCTIONS WITH ALGEBRAIC EXPRESSIONS

Ex 27: For the function $f(x) = 2x + 3$, expand and simplify the expression for $f(x+1)$.

$$f(x+1) = \boxed{2x + 5}$$

Answer: To find the expression for $f(x+1)$, we substitute every instance of x in the formula for $f(x)$ with the expression $(x+1)$.

$$\begin{aligned} f(x+1) &= 2(x+1) + 3 \\ &= 2x + 2 + 3 \\ &= 2x + 5 \end{aligned}$$

Ex 28: For the function $f(x) = x^2 - 1$, expand and simplify the expression for $f(x-1)$.

$$f(x-1) = \boxed{x^2 - 2x}$$

Answer: To find the expression for $f(x-1)$, we substitute every instance of x in the formula for $f(x)$ with the expression $(x-1)$.

$$\begin{aligned} f(x-1) &= (x-1)^2 - 1 \\ &= (x^2 - 2x + 1) - 1 \\ &= x^2 - 2x \end{aligned}$$

Ex 29: For the function $f(x) = 10 - 3x$, expand and simplify the expression for $f(x+2)$.

$$f(x+2) = \boxed{4 - 3x}$$

Answer: To find the expression for $f(x+2)$, we substitute every instance of x in the formula for $f(x)$ with the expression $(x+2)$.

$$\begin{aligned} f(x+2) &= 10 - 3(x+2) \\ &= 10 - 3x - 6 \\ &= 4 - 3x \end{aligned}$$

Ex 30: For the function $f(x) = x^2 - 1$, expand and simplify the expression for $f(x^2+1)$.

$$f(x^2+1) = \boxed{x^4 + 2x^2}$$

Answer: To find the expression for $f(x^2+1)$, we substitute every instance of x in the formula for $f(x)$ with the expression (x^2+1).

$$\begin{aligned} f(x^2+1) &= (x^2+1)^2 - 1 \\ &= ((x^2)^2 + 2(x^2)(1) + 1^2) - 1 \\ &= (x^4 + 2x^2 + 1) - 1 \\ &= x^4 + 2x^2 \end{aligned}$$

A.1.6 SUBSTITUTING VALUES AND EXPRESSIONS INTO A FUNCTION

Ex 31: For $f : x \mapsto 1 - 3x$, find in simplest form:

1. $f(-2) = \boxed{7}$
2. $f(3) = \boxed{-8}$
3. $f(x+1) = \boxed{-3x - 2}$
4. $f(x^2) = \boxed{1 - 3x^2}$

Answer:

1. $\begin{aligned} f(-2) &= 1 - 3 \times (-2) \quad &\text{(substituting } x \text{ with } -2) \\ &= 1 + 6 \\ &= 7 \end{aligned}$
2. $\begin{aligned} f(3) &= 1 - 3 \times 3 \quad &\text{(substituting } x \text{ with } 3) \\ &= 1 - 9 \\ &= -8 \end{aligned}$
3. $\begin{aligned} f(x+1) &= 1 - 3(x+1) \quad &\text{(substituting } x \text{ with } (x+1)) \\ &= 1 - 3x - 3 &\text{(expand)} \\ &= -3x - 2 \end{aligned}$
4. $\begin{aligned} f(x^2) &= 1 - 3(x^2) \quad &\text{(substituting } x \text{ with } (x^2)) \\ &= 1 - 3x^2 \end{aligned}$

Ex 32: For $f : x \mapsto x^2$, find in simplest form:

1. $f(3) = \boxed{9}$
2. $f(-1) = \boxed{1}$
3. $f(-x) = \boxed{x^2}$
4. $f(x+1) = \boxed{x^2 + 2x + 1}$
5. $f(x+2) = \boxed{x^2 + 4x + 4}$
6. $f(2x) = \boxed{4x^2}$

Answer:

1. $f(3) = 3^2 = 9$
2. $f(-1) = (-1)^2 = 1$
3. $\begin{aligned} f(-x) &= (-x)^2 \quad &\text{(substituting } x \text{ with } (-x)) \\ &= (-1)^2 x^2 \\ &= x^2 \end{aligned}$
4. $\begin{aligned} f(x+1) &= (x+1)^2 \quad &\text{(substituting } x \text{ with } (x+1)) \\ &= x^2 + 2x + 1 &\text{(binomial expansion)} \end{aligned}$
5. $\begin{aligned} f(x+2) &= (x+2)^2 \quad &\text{(substituting } x \text{ with } (x+2)) \\ &= x^2 + 4x + 4 &\text{(binomial expansion)} \end{aligned}$
6. $\begin{aligned} f(2x) &= (2x)^2 \quad &\text{(substituting } x \text{ with } (2x)) \\ &= 4x^2 \end{aligned}$

Ex 33: For $g : x \mapsto x^2 - 2x + 1$, find in simplest form:

1. $g(3) = \boxed{4}$
2. $g(-1) = \boxed{4}$
3. $g(-x) = \boxed{x^2 + 2x + 1}$
4. $g(x+1) = \boxed{x^2}$
5. $g(x+2) = \boxed{x^2 + 2x + 1}$
6. $g(2x) = \boxed{4x^2 - 4x + 1}$

Answer:

1. $\begin{aligned} g(3) &= (3)^2 - 2 \times (3) + 1 \quad &\text{(substituting } x \text{ with } 3) \\ &= 9 - 6 + 1 &\text{(evaluate)} \\ &= 4 \end{aligned}$
2. $\begin{aligned} g(-1) &= (-1)^2 - 2 \times (-1) + 1 \quad &\text{(substituting } x \text{ with } -1) \\ &= 1 + 2 + 1 &\text{(evaluate)} \\ &= 4 \end{aligned}$
3. $\begin{aligned} g(-x) &= (-x)^2 - 2 \times (-x) + 1 \quad &\text{(substituting } x \text{ with } (-x)) \\ &= x^2 + 2x + 1 &\text{(expand)} \end{aligned}$
4. $\begin{aligned} g(x+1) &= (x+1)^2 - 2(x+1) + 1 \quad &\text{(substituting } x \text{ with } (x+1 \\ &= (x^2 + 2x + 1) - (2x + 2) + 1 &\text{(expand)} \\ &= x^2 + 2x + 1 - 2x - 2 + 1 &\text{(combine)} \\ &= x^2 \end{aligned}$

5. $$\begin{aligned} g(x+2) &= (x+2)^2 - 2(x+2) + 1 && \text{(substituting } x \text{ with } (x+2)) \\ &= (x^2+4x+4) - (2x+4) + 1 && \text{(expand)} \\ &= x^2+4x+4-2x-4+1 && \text{(combine)} \\ &= x^2+2x+1 \end{aligned}$$

6. $$\begin{aligned} g(2x) &= (2x)^2 - 2 \times (2x) + 1 && \text{(substituting } x \text{ with } (2x)) \\ &= 4x^2 - 4x + 1 && \text{(expand)} \end{aligned}$$

A.1.7 SOLVING LINEAR EQUATIONS FOR $f(x) = y$

Ex 34: Let $f(x) = 3x + 12$. Find all x such that $f(x) = 0$. Justify your answer.

Answer: We solve the equation:

$$\begin{aligned} f(x) &= 0 \\ 3x + 12 &= 0 \\ 3x &= -12 && \text{(subtract 12 from both sides)} \\ x &= -4 && \text{(divide both sides by 3)} \end{aligned}$$

So the solution is $x = -4$.
(Optional) We can check this by calculating $f(-4)$:

$$\begin{aligned} f(-4) &= 3 \times (-4) + 12 \\ &= -12 + 12 \\ &= 0 \end{aligned}$$

Ex 35: Let $f(x) = 2x - 18$. Find all x such that $f(x) = 0$. Justify your answer.

Answer: We solve the equation:

$$\begin{aligned} f(x) &= 0 \\ 2x - 18 &= 0 \\ 2x - 18 + 18 &= 0 + 18 && \text{(add 18 to both sides)} \\ 2x &= 18 \\ \frac{2x}{2} &= \frac{18}{2} && \text{(divide both sides by 2)} \\ x &= 9 \end{aligned}$$

So the solution is $x = 9$.
(Optional) We can check this by calculating $f(9)$:

$$\begin{aligned} f(9) &= 2 \times 9 - 18 \\ &= 18 - 18 \\ &= 0 \end{aligned}$$

Ex 36: Let $f(x) = 2x + 20$. Find all x such that $f(x) = 10$. Justify your answer.

Answer: We solve the equation:

$$\begin{aligned} f(x) &= 10 \\ 2x + 20 &= 10 \\ 2x + 20 - 20 &= 10 - 20 && \text{(subtract 20 from both sides)} \\ 2x &= -10 \\ \frac{2x}{2} &= \frac{-10}{2} && \text{(divide both sides by 2)} \\ x &= -5 \end{aligned}$$

So the solution is $x = -5$.
(Optional) We can check this by calculating $f(-5)$:

$$\begin{aligned} f(-5) &= 2 \times (-5) + 20 \\ &= -10 + 20 \\ &= 10 \end{aligned}$$

Ex 37: Let $f(x) = -6x + 7$. Find all x such that $f(x) = 2$. Justify your answer.

Answer: We solve the equation:

$$\begin{aligned} f(x) &= 2 \\ -6x + 7 &= 2 \\ -6x + 7 - 7 &= 2 - 7 && \text{(subtract 7 from both sides)} \\ -6x &= -5 \\ \frac{-6x}{-6} &= \frac{-5}{-6} && \text{(divide both sides by } -6) \\ x &= \frac{5}{6} \end{aligned}$$

So the solution is $x = \frac{5}{6}$.
(Optional) We can check this by calculating $f\left(\frac{5}{6}\right)$:

$$\begin{aligned} f\left(\frac{5}{6}\right) &= -6 \times \frac{5}{6} + 7 \\ &= -5 + 7 \\ &= 2 \end{aligned}$$

A.1.8 FINDING PREIMAGES

Ex 38: Let $f : x \mapsto \dfrac{4x+1}{x-2}$. Find the value of x for which $f(x) = 3$. Justify your answer.

Answer: We solve the equation:

$$\begin{aligned} f(x) &= 3 \\ \frac{4x+1}{x-2} &= 3 \\ 4x + 1 &= 3(x-2) && \text{(multiply both sides by } (x-2)) \\ 4x + 1 &= 3x - 6 && \text{(expand the right side)} \\ 4x - 3x &= -6 - 1 && \text{(rearrange terms)} \\ x &= -7 \end{aligned}$$

The preimage of 3 is $\boldsymbol{x = -7}$.

Ex 39: Let $f : x \mapsto \sqrt{2x+5}$. Find the value of x such that $f(x) = 3$. Justify your answer.

Answer: We solve the equation:

$$\begin{aligned} f(x) &= 3 \\ \sqrt{2x+5} &= 3 \\ (\sqrt{2x+5})^2 &= 3^2 && \text{(square both sides)} \\ 2x + 5 &= 9 \\ 2x &= 4 && \text{(subtract 5 from both sides)} \\ x &= 2 && \text{(divide by 2)} \end{aligned}$$

The preimage of 3 is $\boldsymbol{x = 2}$. We can check that this value is in the natural domain of f (where $2x + 5 \geq 0$). For $x = 2$, we have $2(2) + 5 = 9 \geq 0$, so the solution is valid.

Ex 40: Let $f : x \mapsto x^2 - 6x + 8$. Find all x such that $f(x) = 0$. Justify your answer.

Answer: We need to find the value(s) of x for which $f(x) = 0$. This requires solving a quadratic equation.

$$\begin{aligned} f(x) &= 0 \\ x^2 - 6x + 8 &= 0 \\ (x-2)(x-4) &= 0 \quad \text{(factor the quadratic)} \end{aligned}$$

This gives two possible solutions:

$$x - 2 = 0 \Leftrightarrow x = 2$$

$$x - 4 = 0 \Leftrightarrow x = 4$$

The preimages of 0 are $\boldsymbol{x = 2}$ and $\boldsymbol{x = 4}$.
Note: If factoring is not straightforward, the roots of any quadratic equation $ax^2 + bx + c = 0$ can be found using the quadratic formula after calculating the discriminant $\Delta = b^2 - 4ac$. For the equation $x^2 - 6x + 8 = 0$, we have $a = 1$, $b = -6$, and $c = 8$.

1. **Calculate the discriminant**:

$$\Delta = (-6)^2 - 4(1)(8) = 36 - 32 = 4$$

2. **Apply the quadratic formula**: Since $\Delta > 0$, there are two distinct real roots.

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-6) \pm \sqrt{4}}{2(1)} = \frac{6 \pm 2}{2}$$

This gives the same two solutions:

$$x_1 = \frac{6+2}{2} = \frac{8}{2} = 4$$

$$x_2 = \frac{6-2}{2} = \frac{4}{2} = 2$$

Ex 41: Let $f(x) = x^2 - 2x + 5$. Find all real numbers x such that $f(x) = 1$. Justify your answer.

Answer: We need to find the value(s) of x for which $f(x) = 1$. We set up the equation:

$$\begin{aligned} f(x) &= 1 \\ x^2 - 2x + 5 &= 1 \\ x^2 - 2x + 4 &= 0 \quad \text{(rearrange into standard quadratic form)} \end{aligned}$$

To solve this quadratic equation, we first calculate the discriminant, $\Delta = b^2 - 4ac$, where $a = 1$, $b = -2$, and $c = 4$.

$$\Delta = (-2)^2 - 4(1)(4) = 4 - 16 = -12$$

Since the discriminant is negative ($\Delta < 0$), the quadratic equation has no real solutions.
Therefore, there are **no real preimages** of 1 under the function f.

A.1.9 ANALYZING LINEAR MODELS IN CONTEXT

Ex 42: The value of a laptop t years after purchase is given by $V(t) = 1800 - 300t$ dollars.

1. Find $V(3)$

   900

   State what this value means
   The value of the laptop after 3 years is \$900.

2. Find t when $V(t) = 600$.

   4

   Explain what this represents.
   After 4 years, the laptop is worth \$600.

3. Find the original purchase price of the laptop.

   1800

Answer:

1. $V(3) = 1800 - 300 \times 3 = 1800 - 900 = 900$.
   This means the value of the laptop after 3 years is \$900.

2. Solve $1800 - 300t = 600$:

$$\begin{aligned} 1800 - 300t &= 600 \\ 1800 - 600 &= 300t \\ 1200 &= 300t \\ t &= 4. \end{aligned}$$

   This represents that after 4 years, the laptop is worth \$600.

3. The original purchase price is $V(0) = 1800 - 300 \times 0 = 1800$ dollars.

Ex 43: The height of a plant t weeks after planting is given by $H(t) = 5 + 2t$ cm.

1. Find $H(4)$

   13

   State what this value means
   The height of the plant after 4 weeks is 13 cm.

2. Find t when $H(t) = 15$.

   5

   Explain what this represents.
   After 5 weeks, the plant is 15 cm tall.

3. Find the initial height of the plant.

   5

Answer:

1. $H(4) = 5 + 2 \times 4 = 5 + 8 = 13$.
   This means the height of the plant after 4 weeks is 13 cm.

2. Solve $5 + 2t = 15$:

$$\begin{aligned} 5 + 2t &= 15 \\ 2t &= 10 \\ t &= 5. \end{aligned}$$

   This represents that after 5 weeks, the plant is 15 cm tall.

3. The initial height is $H(0) = 5 + 2 \times 0 = 5$ cm.

Ex 44: The temperature of water t minutes after starting to heat it is given by $T(t) = 25 + 15t$ degrees Celsius.

1. Find $T(3)$

$$\boxed{70}$$

State what this value means
The temperature of the water after 3 minutes is 70°C.

2. Find t when $T(t) = 100$.

$$\boxed{5}$$

Explain what this represents.
After 5 minutes, the water reaches boiling point at 100°C.

3. Find the initial temperature of the water.

$$\boxed{25}$$

Answer:

1. $T(3) = 25 + 15 \times 3 = 25 + 45 = 70$.
This means the temperature of the water after 3 minutes is 70°C.

2. Solve $25 + 15t = 100$:

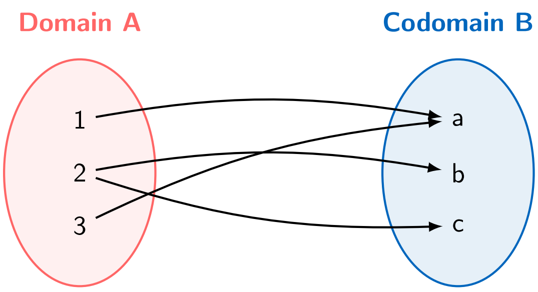
$$\begin{aligned} 25 + 15t &= 100 \\ 15t &= 75 \\ t &= 5. \end{aligned}$$

This represents that after 5 minutes, the water reaches boiling point at 100°C.

3. The initial temperature is $T(0) = 25 + 15 \times 0 = 25$°C.

A.1.10 IDENTIFYING FUNCTIONS FROM MAPPINGS

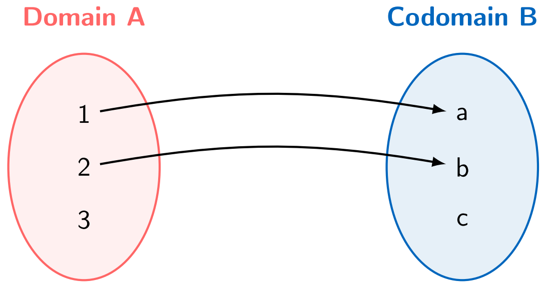
Ex 45: A rule f maps elements from the set $A = \{1, 2, 3\}$ to the set $B = \{a, b, c\}$. The mappings are shown in the diagram below.



Is f a function? Explain your reasoning.

Answer: No, f is not a function. A function must assign **exactly one** output to each input. In this case, the input element 2 from the domain is mapped to two different outputs, b and c. This violates the definition of a function.

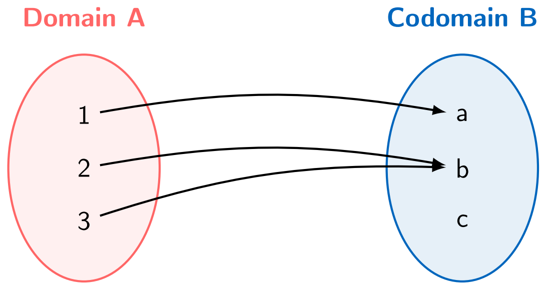
Ex 46: A rule g maps elements from the set $A = \{1, 2, 3\}$ to the set $B = \{a, b, c\}$. The mappings are shown in the diagram below.



Is g a function? Explain your reasoning.

Answer: No, g is not a function. A function must assign an output to **every** element in the domain. In this case, the input element 3 from the domain is not mapped to any output.

Ex 47: A rule h maps elements from the set $A = \{1, 2, 3\}$ to the set $B = \{a, b, c\}$. The mappings are shown in the diagram below.



Is h a function? Explain your reasoning.

Answer: Yes, h is a function. Every element in the domain A (1, 2, and 3) is mapped to exactly one element in the codomain B. The fact that inputs 2 and 3 both map to the same output b is allowed. The function is defined by:

$$h(1) = a, h(2) = b, h(3) = b$$

A.1.11 DOMAIN, CODOMAIN, AND NOTATION

Ex 48: Consider the function defined as $f : \begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ x & \longmapsto & x - 5 \end{array}$.

1. What is the domain of f?
2. What is the codomain of f?
3. What is the image of $x = 7$?
4. What is the preimage of $y = -3$?

Answer:

1. The domain is the set of all integers, $\mathbb{Z}$.
2. The codomain is the set of all integers, $\mathbb{Z}$.
3. The image of $x = 7$ is $f(7) = 7 - 5 = 2$.
4. To find the preimage of $y = -3$, we solve $f(x) = -3$:

$$\begin{aligned} x - 5 &= -3 \\ x &= -3 + 5 \\ x &= 2 \end{aligned}$$

The preimage of -3 is 2.

Ex 49: A function g has the domain $\mathbb{N} = \{1, 2, 3, ...\}$ and codomain $\mathbb{N}$. The rule is "divide the input by 2".

1. Write the function using formal notation.
2. Explain why this rule does not define a valid function $g : \mathbb{N} \to \mathbb{N}$.

Answer:

1. The formal notation is $g : \begin{array}{rcl} \mathbb{N} & \longrightarrow & \mathbb{N} \\ x & \longmapsto & \dfrac{x}{2} \end{array}$.
2. This is not a valid function because the rule does not produce an output in the codomain for every input from the domain. For example, if we take the input $x = 3$ from the domain $\mathbb{N}$, the output is $g(3) = \frac{3}{2}$. This output is not in the codomain $\mathbb{N}$. A function must map every element of its domain to an element within its codomain.

Ex 50: Consider the function defined as $f : \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^2 + 1 \end{array}$.

1. What is the domain of f?
2. What is the codomain of f?
3. What is the image of $x = -3$?
4. What are the preimage(s) of $y = 5$?

Answer:

1. The domain is the set of all real numbers, $\mathbb{R}$.
2. The codomain is the set of all real numbers, $\mathbb{R}$.
3. The image of $x = -3$ is $f(-3) = (-3)^2 + 1 = 9 + 1 = 10$.
4. To find the preimage(s) of $y = 5$, we solve $f(x) = 5$:

$$\begin{aligned} x^2 + 1 &= 5 \\ x^2 &= 4 \\ x &= \pm\sqrt{4} \\ x = 2 &\text{ or } x = -2 \end{aligned}$$

   The preimages of 5 are 2 and -2.

Ex 51: A rule h is defined by $h : \begin{array}{rcl} \mathbb{Z} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \sqrt{x} \end{array}$.

1. State the domain and codomain of h.
2. Explain why this rule does not define a valid function.

Answer:

1. The domain is the set of all integers, $\mathbb{Z}$. The codomain is the set of all real numbers, $\mathbb{R}$.
2. This is not a valid function because the rule is not defined for all elements of the domain. The domain $\mathbb{Z}$ includes negative integers. For any negative integer, such as $x = -1$, the output would be $h(-1) = \sqrt{-1}$, which is not a real number and thus not an element of the codomain $\mathbb{R}$.

Ex 52: Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, 1, 2, 3, 4\}$. Consider the function $k : \begin{array}{rcl} A & \longrightarrow & B \\ x & \longmapsto & x^2 \end{array}$.

1. What is the domain of k?
2. What is the codomain of k?
3. Find the image for each element in the domain.

Answer:

1. The domain is the set $A = \{-2, -1, 0, 1, 2\}$.
2. The codomain is the set $B = \{0, 1, 2, 3, 4\}$.
3. We calculate the image for each input:
   - $k(-2) = (-2)^2 = 4$
   - $k(-1) = (-1)^2 = 1$
   - $k(0) = 0^2 = 0$
   - $k(1) = 1^2 = 1$
   - $k(2) = 2^2 = 4$

   All these images $(0, 1, 4)$ are in the codomain B.

A.2 NATURAL DOMAIN AND RANGE

A.2.1 FINDING THE NATURAL DOMAIN: LEVEL 1

MCQ 53: Find the domain of the function $f : x \mapsto x^2$.

☒ $\mathbb{R}$

☐ $\{x \in \mathbb{R} \mid x \neq 0\}$

☐ $[0, +\infty)$

☐ $(-\infty, 0)$

Answer: The function $f(x) = x^2$ is defined for all real numbers because squaring any real number yields a real result. Therefore, the domain is all real numbers, which is $\mathbb{R}$.

MCQ 54: Find the domain of the function $f : x \mapsto \dfrac{1}{x}$.

☐ $\mathbb{R}$

☒ $\{x \in \mathbb{R} \mid x \neq 0\}$

☐ $[0, +\infty)$

☐ $(-\infty, 0)$

Answer: The function $f(x) = \frac{1}{x}$ is undefined at $x = 0$ because division by zero is not allowed. Therefore, the domain is all real numbers except 0, which is $\{x \in \mathbb{R} \mid x \neq 0\}$.

MCQ 55: Find the domain of the function $f : x \mapsto \sqrt{x}$.

☐ $\mathbb{R}$

☐ $\{x \in \mathbb{R} \mid x \neq 0\}$

☒ $[0, +\infty)$

☐ $(-\infty, 0)$

Answer: The function $f(x) = \sqrt{x}$ is undefined for negative real numbers because the square root of a negative number is not real. Therefore, the domain is all non-negative real numbers, which is $[0, +\infty)$.

A.2.2 FINDING THE NATURAL DOMAIN: LEVEL 2

MCQ 56: Find the domain of the function $f : x \mapsto \sqrt{2x-4}$.

- ☐ $\mathbb{R}$
- ☐ $\{x \in \mathbb{R} \mid x \neq 4\}$
- ☒ $[2, +\infty)$
- ☐ $(-\infty, 4]$

Answer: The function $f(x) = \sqrt{2x-4}$ is undefined when the expression inside the square root is negative, i.e., when $2x-4 < 0$. Solving this inequality:

$$\begin{aligned} 2x - 4 &< 0 \\ 2x &< 4 \quad \text{(adding 4 to both sides)} \\ x &< 2 \quad \text{(dividing both sides by 2)} \end{aligned}$$

Therefore, the function is defined for $x \geq 2$, so the domain is $[2, +\infty)$.

MCQ 57: Find the domain of the function $f : x \mapsto \dfrac{x}{x-3}$.

- ☐ $\mathbb{R}$
- ☐ $\{x \in \mathbb{R} \mid x \neq 3 \text{ and } x \neq 0\}$
- ☐ $[3, +\infty)$
- ☐ $(-\infty, 3)$
- ☒ $\{x \in \mathbb{R} \mid x \neq 3\}$

Answer: The function $f(x) = \dfrac{x}{x-3}$ is undefined when the denominator is zero, i.e., when $x - 3 = 0$. Solving this equation:

$$\begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned}$$

Therefore, the function is defined for all real numbers except $x = 3$, so the domain is $\{x \in \mathbb{R} \mid x \neq 3\}$.

MCQ 58: Find the domain of the function $f : x \mapsto \dfrac{1}{x^2-9}$.

- ☐ $\mathbb{R}$
- ☐ $(-3, 3)$
- ☐ $[0, +\infty)$
- ☒ $\{x \in \mathbb{R} \mid x \neq -3 \text{ and } x \neq 3\}$
- ☐ $x > 3$

Answer: The function $f(x) = \dfrac{1}{x^2-9}$ is undefined when the denominator is zero, i.e., when $x^2 - 9 = 0$. Solving this equation:

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ x &= 3 \quad \text{or} \quad x = -3 \end{aligned}$$

Therefore, the function is defined for all real numbers except $x = 3$ and $x = -3$, so the domain is $\{x \in \mathbb{R} \mid x \neq 3 \text{ and } x \neq -3\}$.

MCQ 59: Find the domain of the function $f : x \mapsto \sqrt{6-2x}$.

- ☐ $\mathbb{R}$
- ☒ $(-\infty, 3]$
- ☐ $[3, +\infty)$
- ☐ $(-\infty, 6]$

Answer: The function $f(x) = \sqrt{6-2x}$ is undefined when the expression inside the square root is negative, i.e., when $6-2x < 0$. Solving this inequality:

$$\begin{aligned} 6 - 2x &< 0 \\ -2x &< -6 \quad \text{(subtract 6 from both sides)} \\ x &> 3 \quad \text{(divide both sides by -2, reverse the sign)} \end{aligned}$$

Therefore, the function is defined for $x \leq 3$, so the domain is $(-\infty, 3]$.

A.2.3 FINDING THE NATURAL DOMAIN: LEVEL 3

Ex 60: Find the natural domain of the function $f(x) = \dfrac{5}{x+3}$. Express your answer in interval notation.

Answer: The function is rational. Its value is undefined when the denominator is zero.

$$x + 3 = 0 \Leftrightarrow x = -3$$

The natural domain is all real numbers except for -3.
In interval notation, this is $\mathbf{(-\infty, -3) \cup (-3, \infty)}$.

Ex 61: Find the natural domain of the function $g(x) = \sqrt{x-4}$. Express your answer in interval notation.

Answer: The function involves a square root. The expression inside the square root (the radicand) must be non-negative.

$$x - 4 \geq 0 \Leftrightarrow x \geq 4$$

The natural domain is all real numbers greater than or equal to 4.
In interval notation, this is $\mathbf{[4, \infty)}$.

Ex 62: Find the natural domain of the function $h(x) = \dfrac{1}{\sqrt{x-5}}$. Express your answer in interval notation.

Answer: This function has two restrictions:

1. The expression inside the square root must be non-negative: $x - 5 \geq 0$.
2. The denominator cannot be zero: $\sqrt{x-5} \neq 0$.

Combining these two conditions means the expression inside the square root must be strictly positive.

$$x - 5 > 0 \Leftrightarrow x > 5$$

The natural domain is all real numbers strictly greater than 5.
In interval notation, this is $\mathbf{(5, \infty)}$.

Ex 63: Find the natural domain of the function $k(x) = \sqrt{16-x^2}$. Express your answer in interval notation.

Answer: The expression inside the square root must be non-negative.

$$16 - x^2 \geq 0$$

This is a quadratic inequality. We can factor the expression:

$$(4-x)(4+x) \geq 0$$

The roots are $x=4$ and $x=-4$. Since the coefficient of x^2 is negative, the parabola opens downwards. The expression is greater than or equal to zero between its roots.
The natural domain is all real numbers between -4 and 4, inclusive.
In interval notation, this is $\mathbf{[-4,4]}$.

A.2.4 FINDING THE RANGE

Ex 64: Find the range of the function $\begin{array}{rrcl} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & |x|-2 \end{array}$.
Express your answer in interval notation.

Answer: We analyze the components of the function $f(x)=|x|-2$.

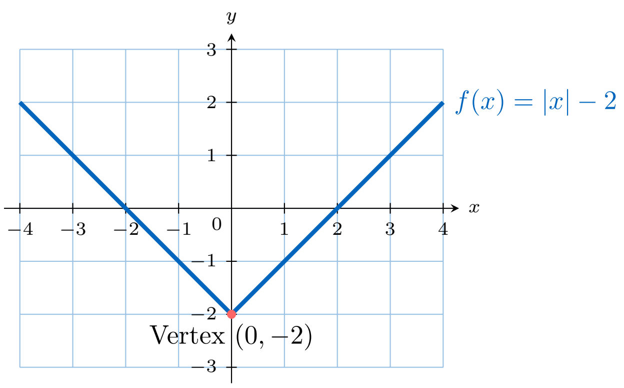
1. The absolute value function, $|x|$, produces only non-negative values, regardless of the input x.

$$|x| \geq 0$$

2. The function $f(x)$ is obtained by subtracting 2 from $|x|$. We can subtract 2 from both sides of the inequality:

$$|x|-2 \geq 0-2$$
$$f(x) \geq -2$$

The minimum value of the function is -2, and it can take any value greater than that.
Therefore, the range is $\mathbf{[-2,\infty)}$.



Ex 65: Find the range of the function $\begin{array}{rrcl} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & (x-2)^2+3 \end{array}$.
Express your answer in interval notation.

Answer: The function is a quadratic in vertex form.

1. The term $(x-2)^2$ is a square, so its minimum value is 0.

$$(x-2)^2 \geq 0$$

2. Adding 3 to both sides of the inequality:

$$(x-2)^2+3 \geq 3$$
$$f(x) \geq 3$$

The minimum value of the function is 3, and it can take any value greater than 3.
The range is $\mathbf{[3,\infty)}$.

Ex 66: Find the range of the function $\begin{array}{rrcl} g: & [0,\infty) & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & 5-\sqrt{x} \end{array}$.
Express your answer in interval notation.

Answer: We analyze the components of the function over its domain $[0,\infty)$.

1. The square root function produces non-negative values:

$$\sqrt{x} \geq 0$$

2. Multiplying by -1 reverses the inequality sign:

$$-\sqrt{x} \leq 0$$

3. Adding 5 to both sides:

$$5-\sqrt{x} \leq 5$$
$$g(x) \leq 5$$

The function's maximum value is 5, and since $\sqrt{x}$ can grow indefinitely, the function can become indefinitely negative.
The range is $\mathbf{(-\infty,5]}$.

A.3 TABLES OF VALUES

A.3.1 FILLING TABLES OF VALUES

Ex 67: For $f(x)=-2x+1$, fill in the table:

x	-2	-1	0	1	2
$f(x)$	5	3	1	-1	-3

Answer:

- $f(-2) = -2\times(-2)+1$ (substituting x with (-2))
  $= 4+1$
  $= 5$
- $f(-1) = -2\times(-1)+1$ (substituting x with (-1))
  $= 2+1$
  $= 3$
- $f(0) = -2\times(0)+1$ (substituting x with (0))
  $= 0+1$
  $= 1$
- $f(1) = -2\times(1)+1$ (substituting x with (1))
  $= -2+1$
  $= -1$
- $f(2) = -2\times(2)+1$ (substituting x with (2))
  $= -4+1$
  $= -3$

So the table of values is:

x	-2	-1	0	1	2
$f(x)$	5	3	1	-1	-3

Ex 68: For $f(x)=x^2-3x+1$, fill in the table:

x	-2	-1	0	1	2
$f(x)$	11	5	1	-1	-1

Answer:

- $f(-2) = ((-2))^2 - 3 \times (-2) + 1$ (substituting x with (-2))
  $= 4 + 6 + 1$
  $= 11$
- $f(-1) = ((-1))^2 - 3 \times (-1) + 1$ (substituting x with (-1))
  $= 1 + 3 + 1$
  $= 5$
- $f(0) = (0)^2 - 3 \times (0) + 1$ (substituting x with (0))
  $= 0 + 0 + 1$
  $= 1$
- $f(1) = (1)^2 - 3 \times (1) + 1$ (substituting x with (1))
  $= 1 - 3 + 1$
  $= -1$
- $f(2) = (2)^2 - 3 \times (2) + 1$ (substituting x with (2))
  $= 4 - 6 + 1$
  $= -1$

So the table of values is:

x	-2	-1	0	1	2
$f(x)$	11	5	1	-1	-1

Ex 69: For the rational function $f(x) = \dfrac{2x}{x+1}$, fill in the table of values.

x	-2	0	1	2
$f(x)$	$\boxed{4}$	$\boxed{0}$	$\boxed{1}$	$\boxed{\frac{4}{3}}$

Answer:

- $f(-2) = \frac{2(-2)}{-2+1} = \frac{-4}{-1} = 4$
- $f(0) = \frac{2(0)}{0+1} = \frac{0}{1} = 0$
- $f(1) = \frac{2(1)}{1+1} = \frac{2}{2} = 1$
- $f(2) = \frac{2(2)}{2+1} = \frac{4}{3}$

The completed table is:

x	-2	0	1	2
$f(x)$	4	0	1	$\frac{4}{3}$

Ex 70: For the absolute value function $g(x) = |x-2|$, fill in the table of values:

x	-1	0	1	2	3
$g(x)$	$\boxed{3}$	$\boxed{2}$	$\boxed{1}$	$\boxed{0}$	$\boxed{1}$

Answer:

- $g(-1) = |-1-2| = |-3| = 3$
- $g(0) = |0-2| = |-2| = 2$
- $g(1) = |1-2| = |-1| = 1$
- $g(2) = |2-2| = |0| = 0$
- $g(3) = |3-2| = |1| = 1$

The completed table is:

x	-1	0	1	2	3
$g(x)$	3	2	1	0	1

A.3.2 FINDING THE FUNCTION FROM A TABLE

Ex 71: The table below gives some values for the function $h(x) = ax + b$. Find the values of a and b and complete the table.

x	0	1	2	5
$h(x)$	-3	$\boxed{-1}$	1	$\boxed{7}$

Answer: We are given two points from the table: $(0, -3)$ and $(2, 1)$.

1. **Find b**: Substitute the point $(0, -3)$ into the function $h(x) = ax + b$.
$$h(0) = a(0) + b = -3 \Leftrightarrow b = -3$$
2. **Find a**: Now we know $h(x) = ax - 3$. Substitute the point $(2, 1)$ into this equation.
$$\begin{aligned} h(2) = a(2) - 3 &= 1 \\ 2a &= 1 + 3 \\ 2a &= 4 \\ a &= 2 \end{aligned}$$

The function is $\boldsymbol{h(x) = 2x - 3}$.
Now we can complete the table:

- $h(1) = 2(1) - 3 = 2 - 3 = -1$
- $h(5) = 2(5) - 3 = 10 - 3 = 7$

The completed table is:

x	0	1	2	5
$h(x)$	-3	-1	1	7

Ex 72: The table below gives some values for the function $f(x) = ax^2 + c$. Find the values of a and c and complete the table.

x	-1	0	2	3
$f(x)$	$\boxed{2}$	-1	11	$\boxed{26}$

Answer: We are given two points from the table: $(0, -1)$ and $(2, 11)$. We can set up a system of two equations to solve for a and c.

1. **Find c**: Substitute the point $(0, -1)$ into the function $f(x) = ax^2 + c$.
$$f(0) = a(0)^2 + c = -1 \Leftrightarrow c = -1$$
2. **Find a**: Now we know $f(x) = ax^2 - 1$. Substitute the point $(2, 11)$ into this equation.
$$\begin{aligned} f(2) = a(2)^2 - 1 &= 11 \\ 4a - 1 &= 11 \\ 4a &= 12 \\ a &= 3 \end{aligned}$$

The function is $\boldsymbol{f(x) = 3x^2 - 1}$.
Now we can complete the table:

- $f(-1) = 3(-1)^2 - 1 = 3(1) - 1 = 2$
- $f(3) = 3(3)^2 - 1 = 3(9) - 1 = 26$

The completed table is:

x	-1	0	2	3
$f(x)$	2	-1	11	26

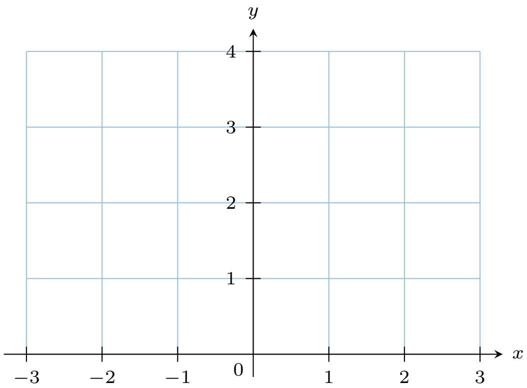
A.4 GRAPHS

A.4.1 PLOTTING LINE GRAPHS

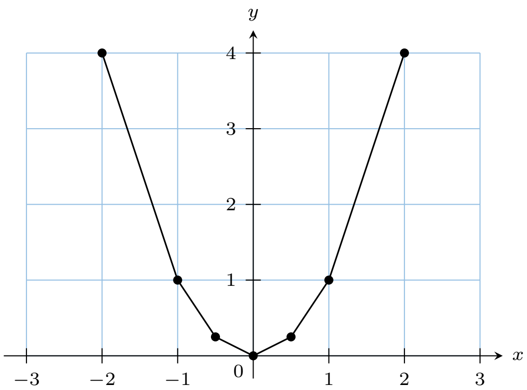
Ex 73: Here is a table of values for the function $f(x) = x^2$:

x	-2	-1	-0.5	0	0.5	1	2
$f(x)$	4	1	0.25	0	0.25	1	4

Plot the line graph of f.



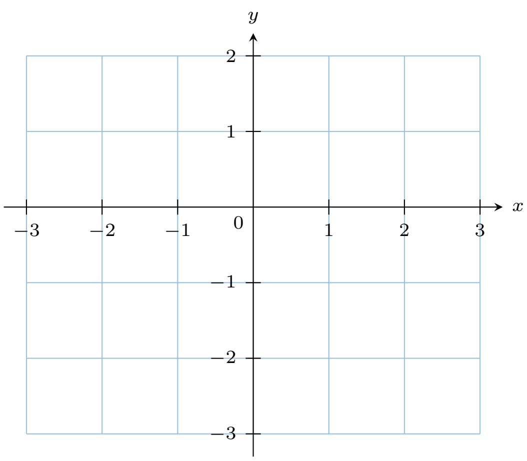
Answer: Plot the points $(-2, 4)$, $(-1, 1)$, $(-0.5, 0.25)$, $(0, 0)$, $(0.5, 0.25)$, $(1, 1)$, and $(2, 4)$. Then, connect the points with straight segments.



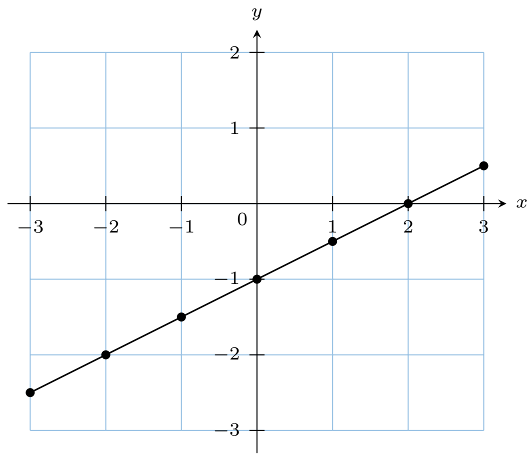
Ex 74: Here is a table of values for the function $f(x) = 0.5x - 1$:

x	-3	-2	-1	0	1	2	3
$f(x)$	-2.5	-2	-1.5	-1	-0.5	0	0.5

Plot the line graph of f.



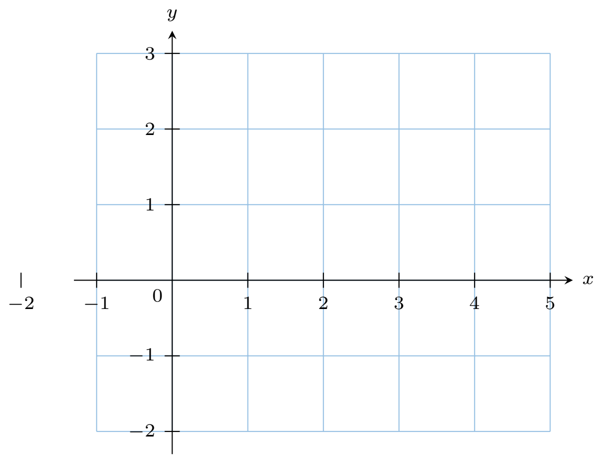
Answer: Plot the points $(-3, -2.5)$, $(-2, -2)$, $(-1, -1.5)$, $(0, -1)$, $(1, -0.5)$, $(2, 0)$, $(3, 0.5)$. Then, connect the points with straight segments to form the line graph.



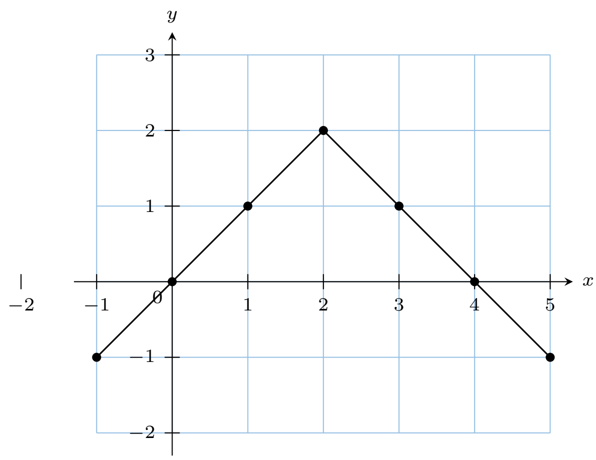
Ex 75: Here is a table of values for the function $f(x) = -|x - 2| + 2$:

x	-1	0	1	2	3	4	5
$f(x)$	-1	0	1	2	1	0	-1

Plot the graph of f.

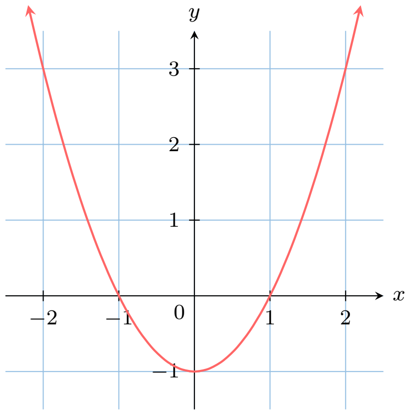


Answer: Plot the points $(-1, -1)$, $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 1)$, $(4, 0)$, and $(5, -1)$. Then, connect the points with straight segments to form the graph.



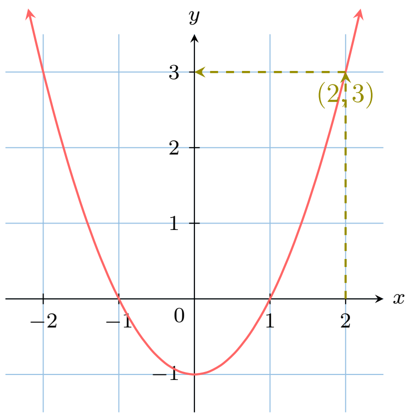
A.4.2 FINDING $f(x)$

Ex 76: The graph of $y = f(x)$ is:



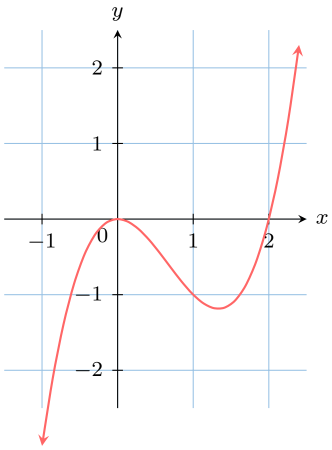
$f(2) = \boxed{3}$

Answer:



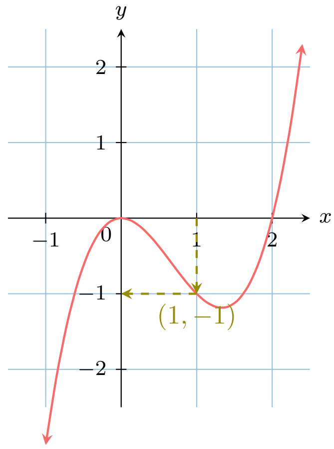
$f(2) = 3$

Ex 77: The graph of $y = f(x)$ is:



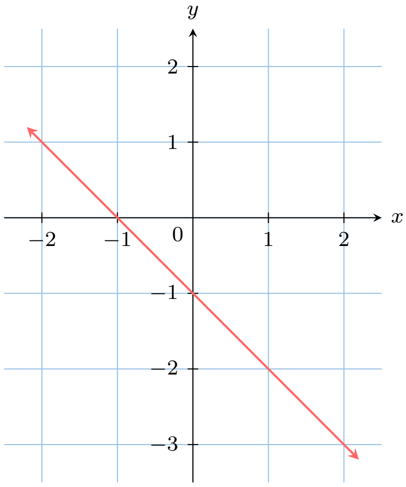
$f(1) = \boxed{-1}$

Answer:



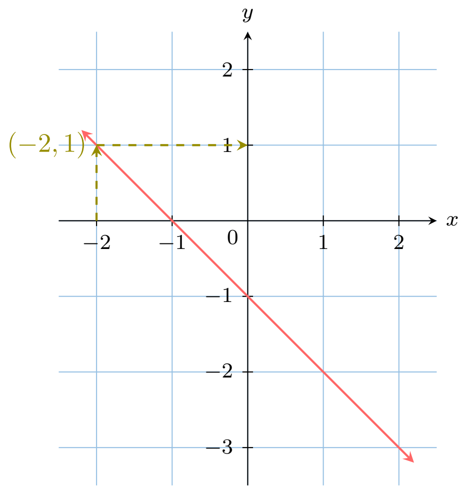
$f(1) = -1$

Ex 78: The graph of $y = f(x)$ is:



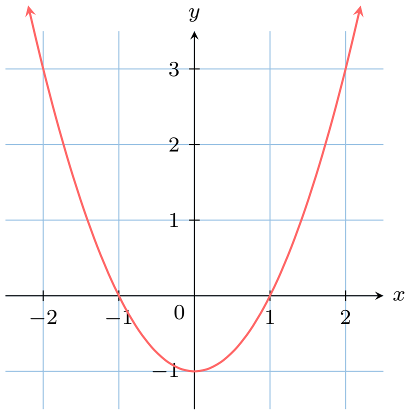
$f(-2) = \boxed{1}$

Answer:



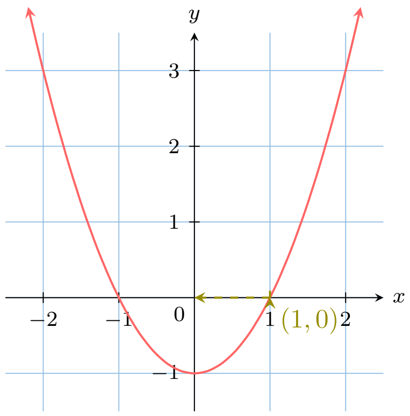
$f(-2) = 1$

Ex 79: The graph of $y = f(x)$ is:



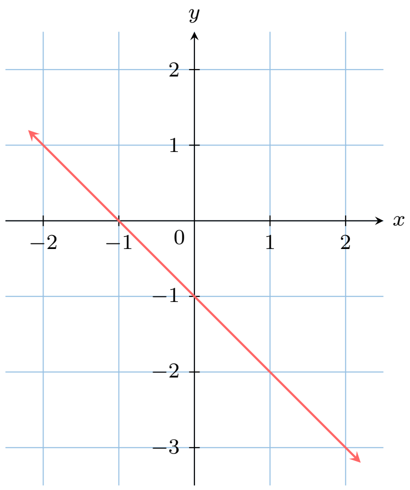
$f(1) = \boxed{0}$

Answer:



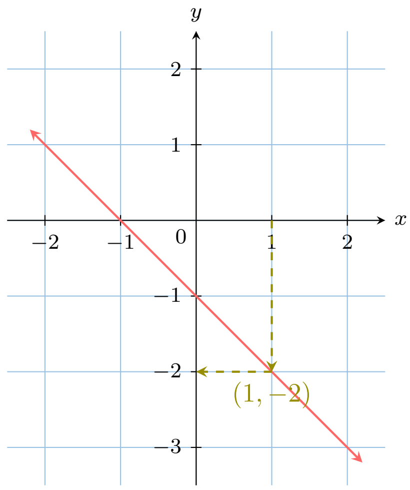
$f(1) = 0$

Ex 80: The graph of $y = f(x)$ is:



$f(1) = \boxed{-2}$

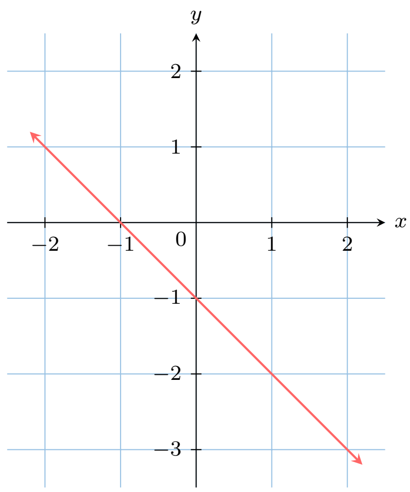
Answer:



$f(1) = -2$

A.4.3 FINDING INPUTS FROM OUTPUTS ON A GRAPH

Ex 81: The graph of $y = f(x)$ is:

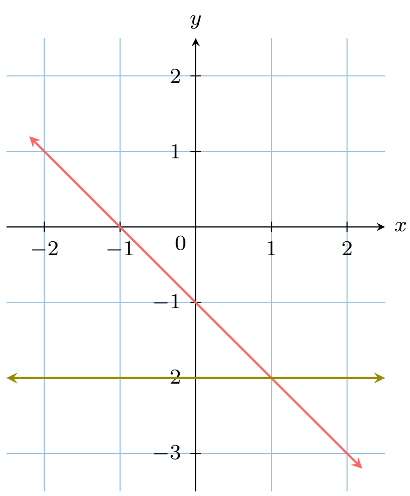


Find all x such that $f(x) = -2$.

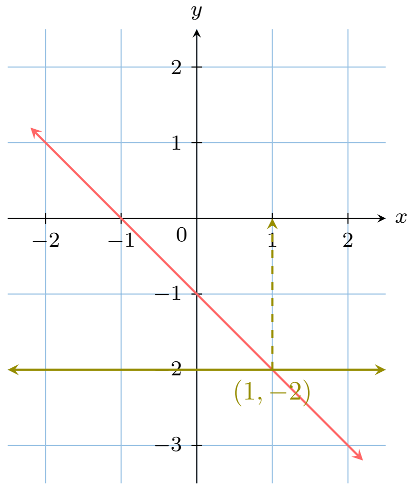
$x = \boxed{1}$

Answer:

- Draw a horizontal line at $y = -2$.

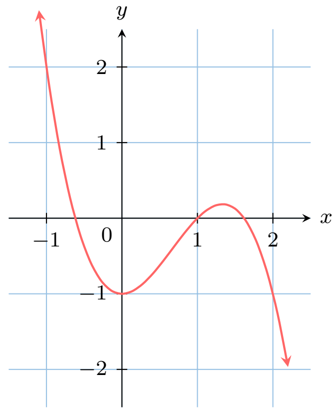


- Identify the intersection point with the curve.



$x = 1$

Ex 82: The graph of $y = f(x)$ is:

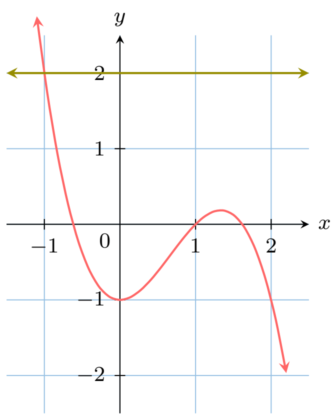


Find all x such that $f(x) = 2$.

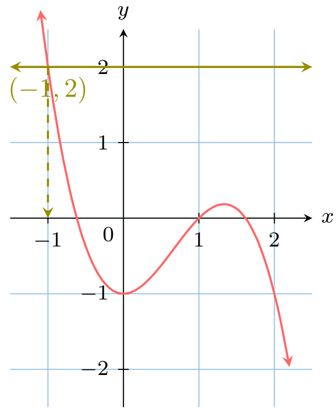
$x = \boxed{-1}$

Answer:

- Draw a horizontal line at $y = 2$.

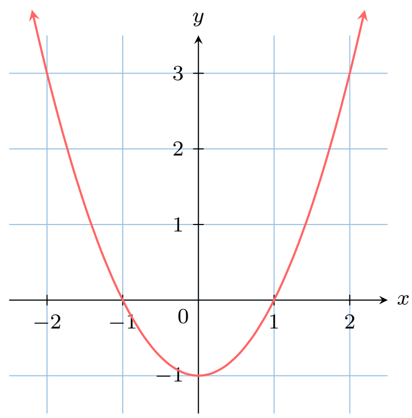


- Identify the intersection point with the curve.



$x = -1$

Ex 83: The graph of $y = f(x)$ is:

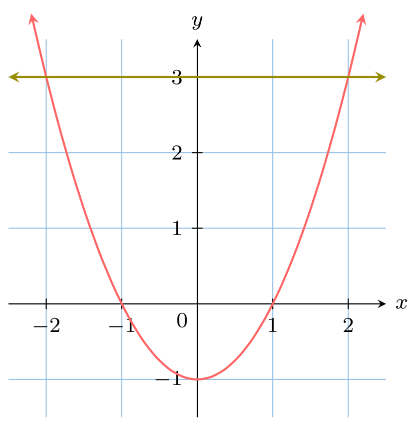


Find all x such that $f(x) = 3$.
Give your answers in increasing order:

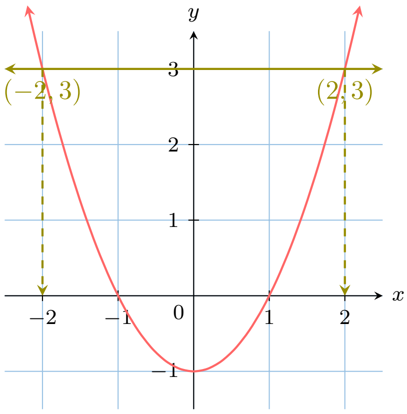
$x = \boxed{-2}$ or $x = \boxed{2}$

Answer:

- Draw a horizontal line at $y = 3$.

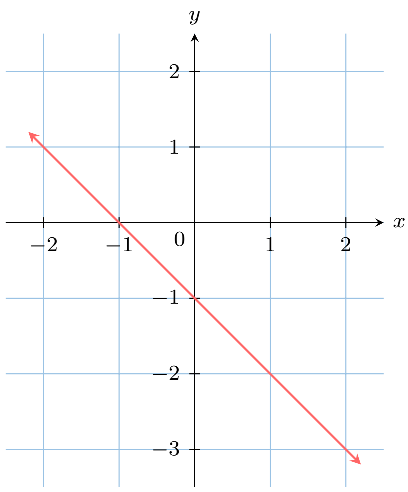


- Identify the intersection points with the curve.



$x = -2$ or $x = 2$

Ex 84: The graph of $y = f(x)$ is:

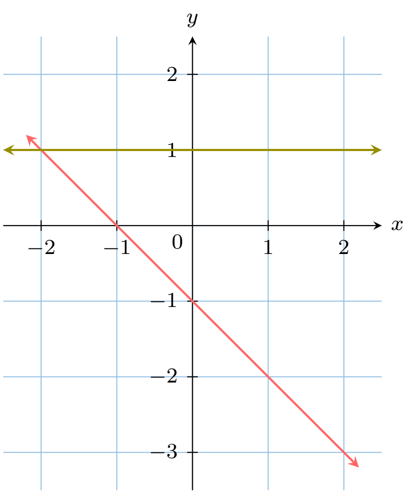


Find all x such that $f(x) = 1$.

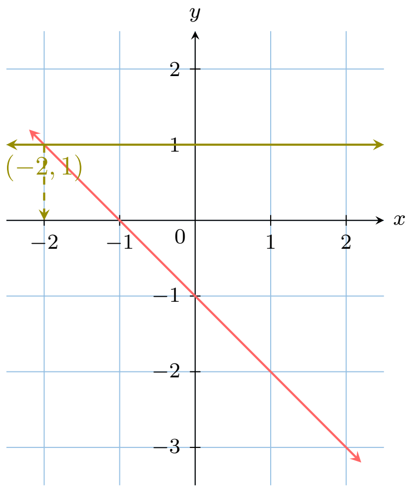
$$x = \boxed{-2}$$

Answer:

- Draw a horizontal line at $y = 1$.



- Identify the intersection point with the curve.

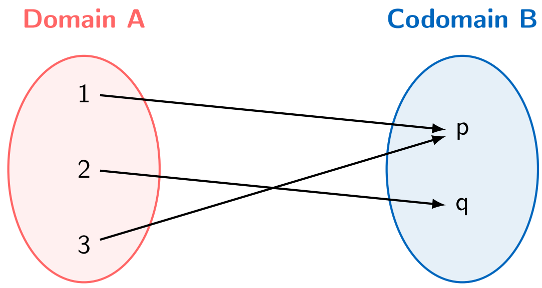


$x = -2$

A.5 BIJECTIVE FUNCTIONS

A.5.1 ANALYZING FUNCTION PROPERTIES FROM MAPPING DIAGRAMS

Ex 85: Let $A = \{1, 2, 3\}$ and $B = \{p, q\}$. A function $f : A \to B$ is defined by the mapping diagram below.

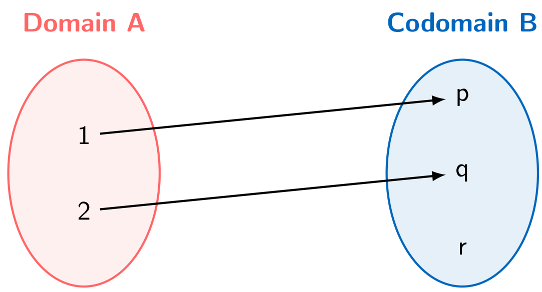


Determine if the function f is injective, surjective, and/or bijective. Justify your answers.

Answer:

- **Injectivity**: The function is **not injective**. Different inputs lead to the same output. Specifically, $f(1) = p$ and $f(3) = p$.
- **Surjectivity**: The function is **surjective**. The range of the function is $\{p, q\}$, which is equal to the codomain B. Every element in B has at least one preimage.
- **Bijectivity**: The function is **not bijective** because it is not injective.

Ex 86: Let $A = \{1, 2\}$ and $B = \{p, q, r\}$. A function $g : A \to B$ is defined by the mapping diagram below.

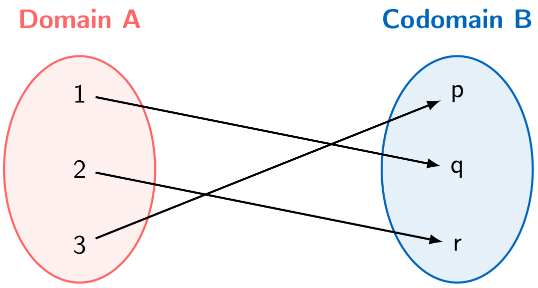


Determine if the function g is injective, surjective, and/or bijective. Justify your answers.

Answer:

- **Injectivity**: The function is **injective**. Every input has a unique output; $g(1) = p$ and $g(2) = q$.
- **Surjectivity**: The function is **not surjective**. The range is $\{p, q\}$, but the codomain is $\{p, q, r\}$. The element r in the codomain has no preimage in the domain.
- **Bijectivity**: The function is **not bijective** because it is not surjective.

Ex 87: Let $A = \{1, 2, 3\}$ and $B = \{p, q, r\}$. A function $h : A \to B$ is defined by the mapping diagram below.

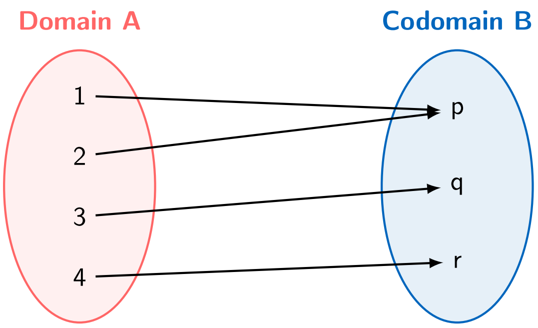


Determine if the function h is injective, surjective, and/or bijective. Justify your answers.

Answer:

- **Injectivity**: The function is **injective**. Every input maps to a distinct output: $h(1) = q$, $h(2) = r$, and $h(3) = p$.
- **Surjectivity**: The function is **surjective**. The range is $\{p, q, r\}$, which is equal to the codomain B. Every element in B has a preimage.
- **Bijectivity**: The function is **bijective** because it is both injective and surjective.

Ex 88: Let $A = \{1, 2, 3, 4\}$ and $B = \{p, q, r\}$. A function $k : A \to B$ is defined by the mapping diagram below.



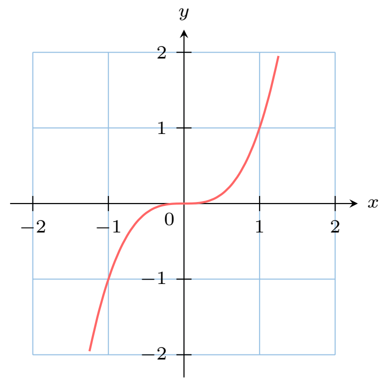
Determine if the function k is injective, surjective, and/or bijective. Justify your answers.

Answer:

- **Injectivity**: The function is **not injective**. The inputs 1 and 2 both map to the same output, p.
- **Surjectivity**: The function is **surjective**. The range is $\{p, q, r\}$, which is the same as the codomain B.
- **Bijectivity**: The function is **not bijective** because it is not injective.

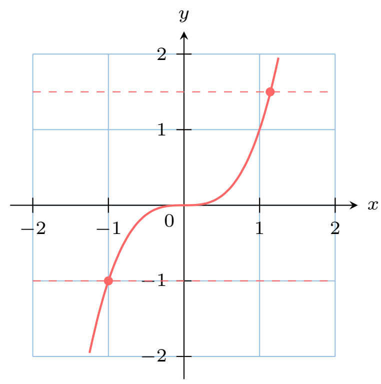
A.5.2 APPLYING THE HORIZONTAL LINE TEST

Ex 89: Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x^3$, graphed below.



Determine if the function f is injective, surjective, and/or bijective. Justify your answers.

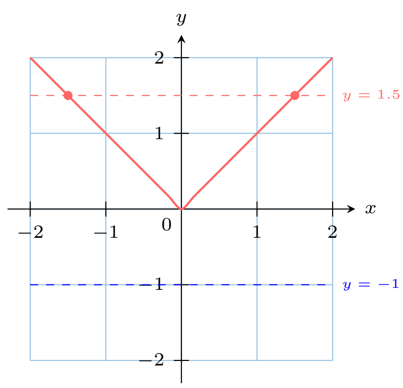
Answer:



Using the horizontal line test, any horizontal line drawn on the graph intersects the curve exactly once.

- **Injective**: Yes, because every horizontal line intersects the graph at most once.
- **Surjective**: Yes, because every horizontal line (for all $y \in \mathbb{R}$) intersects the graph at least once.
- **Bijective**: Yes, because the function is both injective and surjective.

Answer:

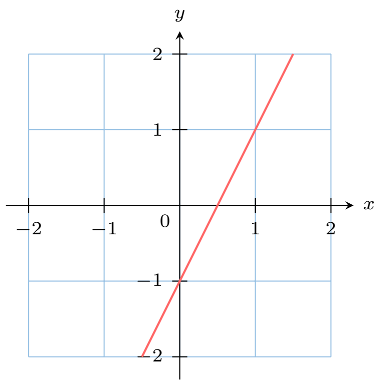


- **Injective**: No. A horizontal line, for example at $y = 1.5$, intersects the graph at two points. This fails the horizontal line test for injectivity.

- **Surjective**: No. The range of the function is $[0, \infty)$, which is not equal to the codomain $\mathbb{R}$. For example, a horizontal line at $y = -1$ (which is in the codomain) does not intersect the graph at all, meaning -1 has no preimage.

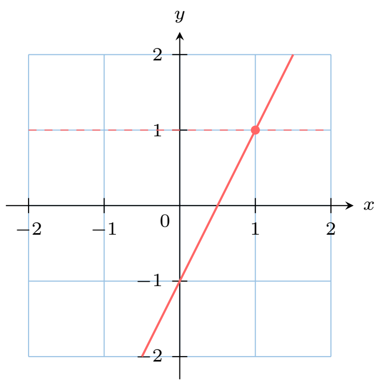
- **Bijective**: No, because the function is neither injective nor surjective.

Ex 90: Consider the function $h : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & 2x-1 \end{array}$, graphed below.



Determine if the function h is injective, surjective, and/or bijective. Justify your answers.

Answer:



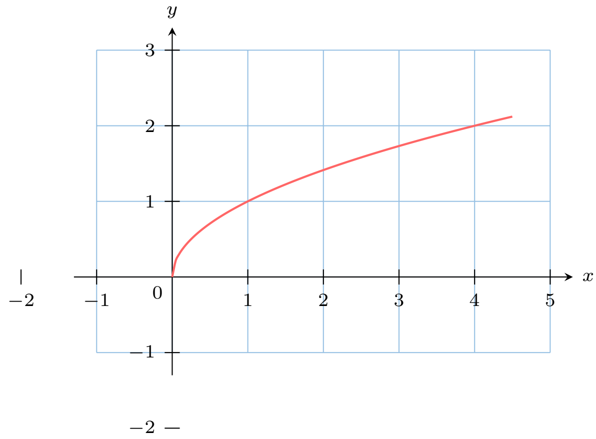
Any horizontal line drawn on the graph will intersect the line at exactly one point.

- **Injective**: Yes, because every horizontal line intersects the graph at most once.

- **Surjective**: Yes, because every horizontal line intersects the graph at least once.

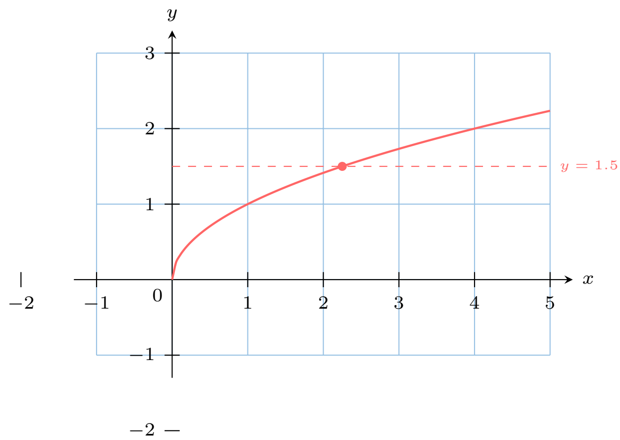
- **Bijective**: Yes, because it is both injective and surjective.

Ex 91: Consider the function $f : \begin{array}{ccc} [0,\infty) & \longrightarrow & [0,\infty) \\ x & \longmapsto & \sqrt{x} \end{array}$, graphed below.



Determine if the function f is injective, surjective, and/or bijective. Justify your answers.

Answer:



We apply the horizontal line test, considering that the codomain is $[0, \infty)$. This means we only need to test horizontal lines where $y \geq 0$.

- **Injective**: Yes. Any horizontal line with $y \geq 0$ intersects the graph at exactly one point. Since it intersects at most once, the function is injective.

- **Surjective**: Yes. Any horizontal line with $y \geq 0$ (i.e., for any value in the codomain) intersects the graph at least once. The range is equal to the codomain.

- **Bijective**: Yes, because the function is both injective and surjective.

B OPERATIONS ON FUNCTIONS

B.1 ALGEBRA OF FUNCTIONS

B.1.1 ADDING, SUBTRACTING, AND MULTIPLYING FUNCTIONS

Ex 92: For $f(x) = 2x + 2$ and $g(x) = 3 - x$, find in simplest form:

1. $f(3) + g(3) = \boxed{8}$
2. $f(-1) + g(-1) = \boxed{4}$
3. $f(x) + g(x) = \boxed{x + 5}$

4. $g(x) + f(x) = \boxed{x+5}$

Answer:

1. $$\begin{aligned} f(3) + g(3) &= (2 \times 3 + 2) + (3 - 3) \\ &= (6+2) + 0 \\ &= 8 + 0 \\ &= 8 \end{aligned}$$
2. $$\begin{aligned} f(-1) + g(-1) &= (2 \times (-1) + 2) + (3 - (-1)) \\ &= (-2+2) + (3+1) \\ &= 0 + 4 \\ &= 4 \end{aligned}$$
3. $$\begin{aligned} f(x) + g(x) &= (2x+2) + (3-x) \\ &= 2x + 2 + 3 - x \\ &= x + 5 \end{aligned}$$
4. $$\begin{aligned} g(x) + f(x) &= (3-x) + (2x+2) \\ &= 3 - x + 2x + 2 \\ &= x + 5 \end{aligned}$$

Ex 93: For $f(x) = x^2 - 2$ and $g(x) = x - 2$, find in simplest form:

1. $f(0) + g(0) = \boxed{-4}$
2. $f(-2) + g(-2) = \boxed{-2}$
3. $f(x) + g(x) = \boxed{x^2 + x - 4}$
4. $f(x) - g(x) = \boxed{x^2 - x}$

Answer:

1. $$\begin{aligned} f(0) + g(0) &= (0^2 - 2) + (0 - 2) \\ &= (-2) + (-2) \\ &= -4 \end{aligned}$$
2. $$\begin{aligned} f(-2) + g(-2) &= ((-2)^2 - 2) + (-2 - 2) \\ &= (4-2) + (-4) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$
3. $$\begin{aligned} f(x) + g(x) &= (x^2 - 2) + (x - 2) \\ &= x^2 + x - 4 \end{aligned}$$
4. $$\begin{aligned} f(x) - g(x) &= (x^2 - 2) - (x - 2) \\ &= x^2 - 2 - x + 2 \\ &= x^2 - x \end{aligned}$$

Ex 94: Let $f(x) = 3x - 2$ and $g(x) = x^2$. Find in factorized form:

$$f(x) \times g(x) = \boxed{(3x-2)x^2}$$

Answer: $$\begin{aligned} f(x) \times g(x) &= (3x-2) \times x^2 \\ &= (3x-2)x^2 \end{aligned}$$

Ex 95: Let $f(x) = 2x + 5$ and $g(x) = x - 4$. Find in factorized form:

$$f(x) \times g(x) = \boxed{(2x+5)(x-4)}$$

Answer: $$\begin{aligned} f(x) \times g(x) &= (2x+5) \times (x-4) \\ &= (2x+5)(x-4) \end{aligned}$$

B.1.2 DECOMPOSING EXPRESSIONS INTO FUNCTIONS

Ex 96: Find two functions f and g such that $f(x) \times g(x) = (x+3)^2(x-2)$.

- $f(x) = \boxed{(x+3)^2}$
- $g(x) = \boxed{x-2}$

Answer: One possible pair is $f(x) = (x+3)^2$ and $g(x) = x - 2$, since

$$f(x) \times g(x) = (x+3)^2 \times (x-2).$$

Ex 97: Find two functions f and g such that $f(x) \times g(x) = (x^2+4)(3x-7)$.

- $f(x) = \boxed{x^2 + 4}$
- $g(x) = \boxed{3x - 7}$

Answer: One possible pair is $f(x) = x^2 + 4$ and $g(x) = 3x - 7$, since

$$f(x) \times g(x) = (x^2+4) \times (3x-7).$$

Ex 98: Find two functions f and g such that $f(x) + g(x) = (x-2)^2 + \sqrt{x}$.

- $f(x) = \boxed{(x-2)^2}$
- $g(x) = \boxed{\sqrt{x}}$

Answer: One possible pair is $f(x) = (x-2)^2$ and $g(x) = \sqrt{x}$, since

$$f(x) + g(x) = (x-2)^2 + \sqrt{x}.$$

Ex 99: Find two functions f and g such that $f(x) + g(x) = \dfrac{1}{x} + (x+1)^2$.

- $f(x) = \boxed{\dfrac{1}{x}}$
- $g(x) = \boxed{(x+1)^2}$

Answer: One possible pair is $f(x) = \dfrac{1}{x}$ and $g(x) = (x+1)^2$, since

$$f(x) + g(x) = \frac{1}{x} + (x+1)^2.$$

B.1.3 OPERATIONS ON FUNCTIONS AND THEIR DOMAINS

Ex 100: Let the functions f and g be defined by the rules $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{3-x}$. Let $h = f + g$.

1. Find the domain of f and the domain of g.
2. Find the domain of the combined function h.
3. Calculate $h(-1)$.

Answer:

1. **Domain of f**: We need $x+2 \geq 0 \Leftrightarrow x \geq -2$. The domain of f is $\mathbf{[-2, \infty)}$.
Domain of g: We need $3-x \geq 0 \Leftrightarrow 3 \geq x$. The domain of g is $\mathbf{(-\infty, 3]}$.

2. The domain of $h = f+g$ is the intersection of the domains of f and g. We need values of x that are both greater than or equal to -2 AND less than or equal to 3. The domain of h is $\mathbf{[-2, 3]}$.

3. To calculate $h(-1)$, we first check that -1 is in the domain of h. Since $-2 \leq -1 \leq 3$, it is valid.

$$\begin{aligned} h(-1) &= f(-1) + g(-1) \\ &= \sqrt{-1+2} + \sqrt{3-(-1)} \\ &= \sqrt{1} + \sqrt{4} \\ &= 1 + 2 = \mathbf{3} \end{aligned}$$

Ex 101: Let the functions f and g be defined by the rules $f(x) = \dfrac{1}{x-4}$ and $g(x) = \sqrt{x-1}$. Let $h = f \times g$.

1. Find the domain of f and the domain of g.

2. Find the domain of the combined function h.

3. Calculate $h(5)$.

Answer:

1. **Domain of f**: The denominator cannot be zero, so $x-4 \neq 0 \Leftrightarrow x \neq 4$. The domain of f is $\mathbb{R} \setminus \mathbf{\{4\}}$.
Domain of g: The expression in the square root must be non-negative, so $x-1 \geq 0 \Leftrightarrow x \geq 1$. The domain of g is $\mathbf{[1, \infty)}$.

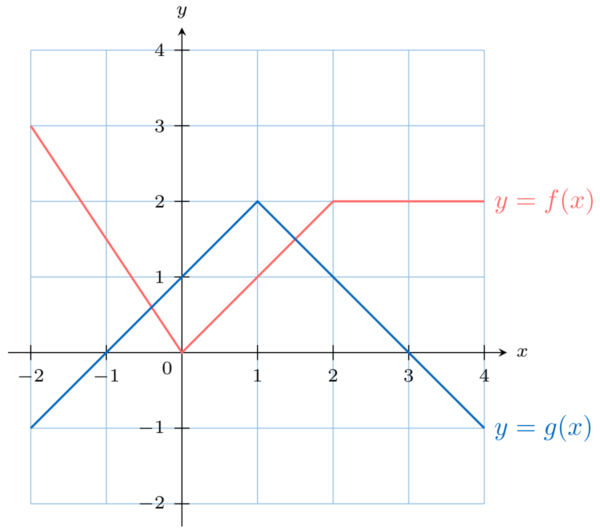
2. The domain of $h = f \cdot g$ is the intersection of the domains of f and g. We need all numbers x such that $x \geq 1$ AND $x \neq 4$. The domain of h is $\mathbf{[1, 4) \cup (4, \infty)}$.

3. To calculate $h(5)$, we first check that 5 is in the domain of h. Since $5 \geq 1$ and $5 \neq 4$, it is valid.

$$\begin{aligned} h(5) &= f(5) \times g(5) \\ &= \left(\frac{1}{5-4}\right) \times \sqrt{5-1} \\ &= \left(\frac{1}{1}\right) \times \sqrt{4} \\ &= 1 \times 2 = \mathbf{2} \end{aligned}$$

B.1.4 GRAPHICAL COMBINATION OF FUNCTIONS

Ex 102: The graphs of two functions, f and g, are shown below.

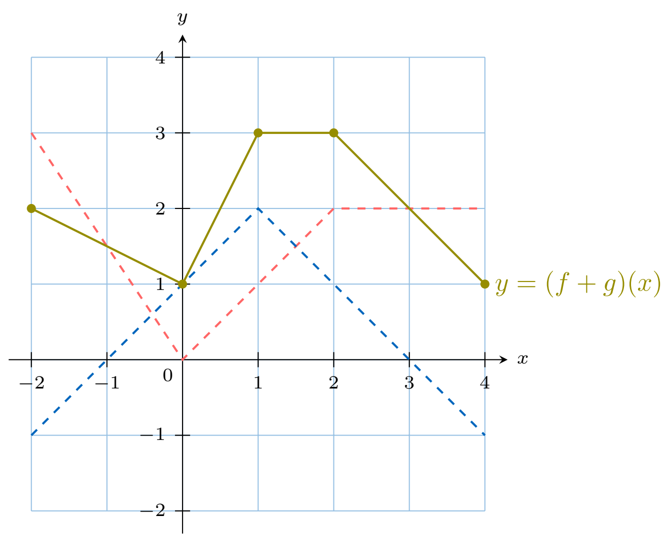


Plot the graph of the function $f+g$.

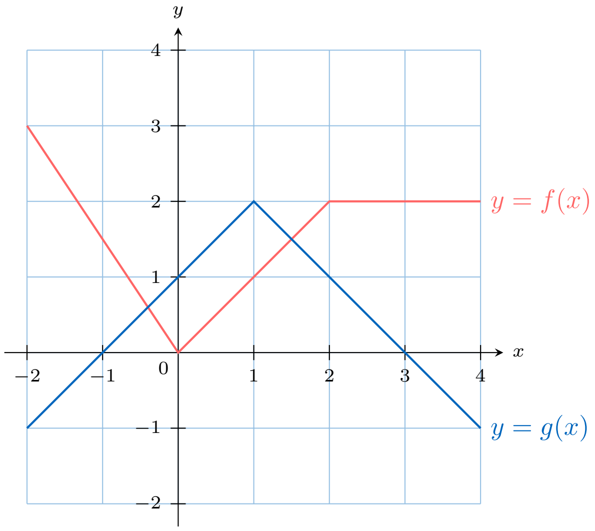
Answer: To plot the graph of $(f+g)(x)$, we add the y-coordinates of the functions f and g at several key x-values.
First, we create a table of values by reading the points from the graphs:

x	-2	0	1	2	4
$f(x)$	3	0	1	2	2
$g(x)$	-1	1	2	1	-1
$(f+g)(x)$	2	1	3	3	1

Now, we plot the new points $(-2, 2)$, $(0, 1)$, $(1, 3)$, $(2, 3)$, and $(4, 1)$ and connect them with line segments.



Ex 103: The graphs of two functions, f and g, are shown below.

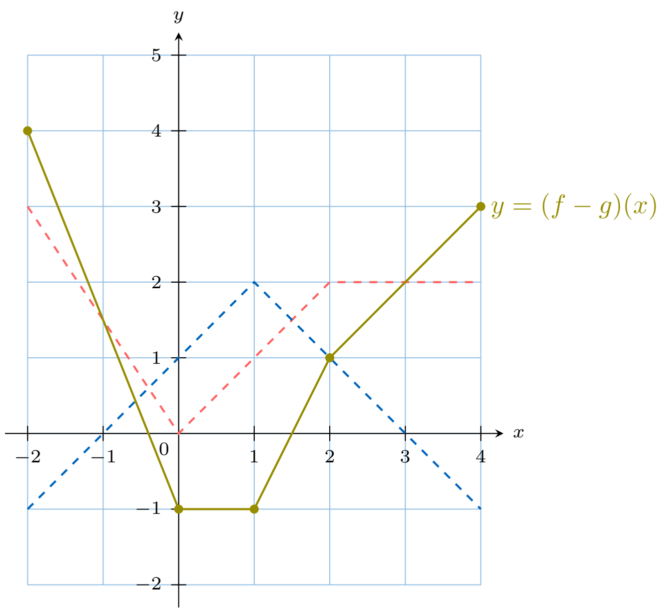


Plot the graph of the function $f - g$.

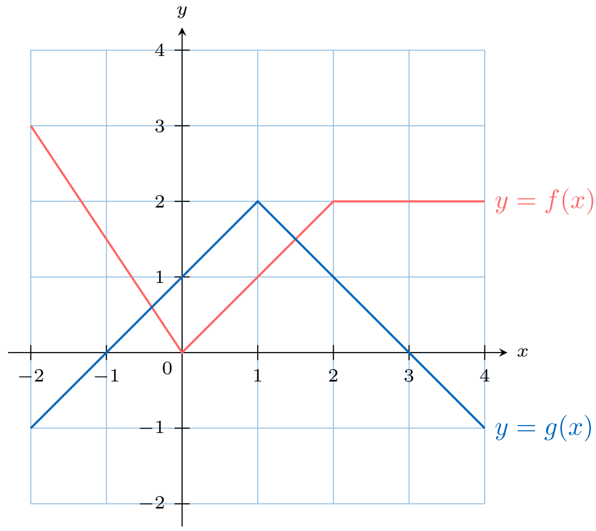
Answer: To plot the graph of $(f-g)(x)$, we subtract the y-coordinates of the function g from the y-coordinates of the function f at several key x-values.
First, we create a table of values by reading the points from the graphs:

x	-2	0	1	2	4
$f(x)$	3	0	1	2	2
$g(x)$	-1	1	2	1	-1
$(f-g)(x)$	4	-1	-1	1	3

Now, we plot the new points $(-2,4)$, $(0,-1)$, $(1,-1)$, $(2,1)$, and $(4,3)$ and connect them with line segments.



Ex 104: The graphs of two functions, f and g, are shown below.

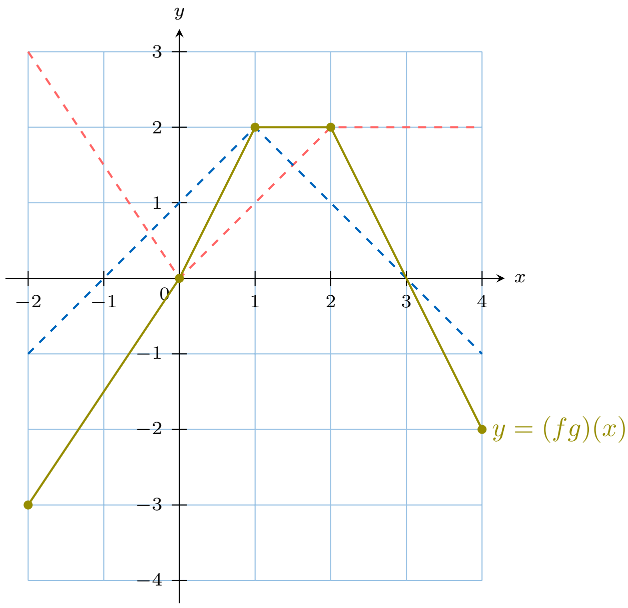


Plot the graph of the function $f \times g$.

Answer: To plot the graph of $(fg)(x)$, we multiply the y-coordinates of the functions f and g at several key x-values.
First, we create a table of values by reading the points from the graphs:

x	-2	0	1	2	4
$f(x)$	3	0	1	2	2
$g(x)$	-1	1	2	1	-1
$(f \times g)(x)$	-3	0	2	2	-2

Now, we plot the new points $(-2,-3)$, $(0,0)$, $(1,2)$, $(2,2)$, and $(4,-2)$ and connect them. Note that since we are multiplying linear functions, the resulting graph between these points will be composed of parabolic curves, not straight lines. We sketch a smooth curve through the points.



B.2 COMPOSITION OF FUNCTIONS

B.2.1 EVALUATING COMPOSITE FUNCTIONS

Ex 105: For $f(x) = 2x + 2$ and $g(x) = 3 - x$, find in simplest form:

1. $f(g(3)) = \boxed{2}$
2. $f(g(-1)) = \boxed{10}$

3. $f(g(x)) = \boxed{8-2x}$

4. $g(f(x)) = \boxed{1-2x}$

Answer:

1. $$\begin{aligned} f(g(3)) &= f(3-3) \\ &= f(0) \\ &= 2\times 0+2 \\ &= 2 \end{aligned}$$

2. $$\begin{aligned} f(g(-1)) &= f(3-(-1)) \\ &= f(4) \\ &= 2\times 4+2 \\ &= 8+2 \\ &= 10 \end{aligned}$$

3. $$\begin{aligned} f(g(x)) &= f(3-x) \\ &= 2(3-x)+2 \\ &= 6-2x+2 \\ &= 8-2x \end{aligned}$$

4. $$\begin{aligned} g(f(x)) &= g(2x+2) \\ &= 3-(2x+2) \\ &= 3-2x-2 \\ &= 1-2x \end{aligned}$$

Ex 106: For $f(x) = x^2+2x$ and $g(x) = 2-x$, find in simplest form:

1. $f(g(3)) = \boxed{-1}$

2. $f(g(-1)) = \boxed{15}$

3. $f(g(x)) = \boxed{x^2-6x+8}$

4. $g(f(x)) = \boxed{2-x^2-2x}$

Answer:

1. $$\begin{aligned} f(g(3)) &= f(2-3) \\ &= f(-1) \\ &= (-1)^2+2\times(-1) \\ &= 1-2 \\ &= -1 \end{aligned}$$

2. $$\begin{aligned} f(g(-1)) &= f(2-(-1)) \\ &= f(3) \\ &= 3^2+2\times 3 \\ &= 9+6 \\ &= 15 \end{aligned}$$

3. $$\begin{aligned} f(g(x)) &= f(2-x) \\ &= (2-x)^2+2(2-x) \\ &= (4-4x+x^2)+(4-2x) \\ &= x^2-6x+8 \end{aligned}$$

4. $$\begin{aligned} g(f(x)) &= g(x^2+2x) \\ &= 2-(x^2+2x) \\ &= 2-x^2-2x \end{aligned}$$

Ex 107: For $f(x) = 3x-5$, find in simplest form:

1. $f(f(-1)) = \boxed{-29}$

2. $f(f(x)) = \boxed{9x-20}$

Answer:

1. $$\begin{aligned} f(f(-1)) &= f(3\times(-1)-5) \\ &= f(-8) \\ &= 3\times(-8)-5 \\ &= -24-5 \\ &= -29 \end{aligned}$$

2. $$\begin{aligned} f(f(x)) &= f(3x-5) && \text{(substituting } x \text{ with } (3x-5)) \\ &= 3(3x-5)-5 \\ &= 9x-15-5 \\ &= 9x-20 \end{aligned}$$

B.2.2 DECOMPOSING FUNCTIONS INTO COMPOSITIONS

Ex 108: Find two functions f and g such that $f(g(x)) = \sqrt{2x-1}$ and $g(x) \neq x$.

- $f(x) = \boxed{\sqrt{x}}$
- $g(x) = \boxed{2x-1}$

Answer: One possible pair is $f(x) = \sqrt{x}$ and $g(x) = 2x-1$, since

$$\begin{aligned} f(g(x)) &= f(2x-1) \\ &= \sqrt{2x-1}. \end{aligned}$$

Ex 109: Find two functions f and g such that $f(g(x)) = (x+2)^5$ and $g(x) \neq x$.

- $f(x) = \boxed{x^5}$
- $g(x) = \boxed{x+2}$

Answer: One possible pair is $f(x) = x^5$ and $g(x) = x+2$, since

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^5. \end{aligned}$$

Ex 110: Find two functions f and g such that $f(g(x)) = \dfrac{1}{x^2+1}$ and $g(x) \neq x$.

- $f(x) = \boxed{\dfrac{1}{x}}$
- $g(x) = \boxed{x^2+1}$

Answer: One possible pair is $f(x) = \dfrac{1}{x}$ and $g(x) = x^2+1$, since

$$\begin{aligned} f(g(x)) &= f(x^2+1) \\ &= \frac{1}{x^2+1}. \end{aligned}$$

Ex 111: Find two functions f and g such that $f(g(x)) = (x^3-2)^{-4}$ and $g(x) \neq x$.

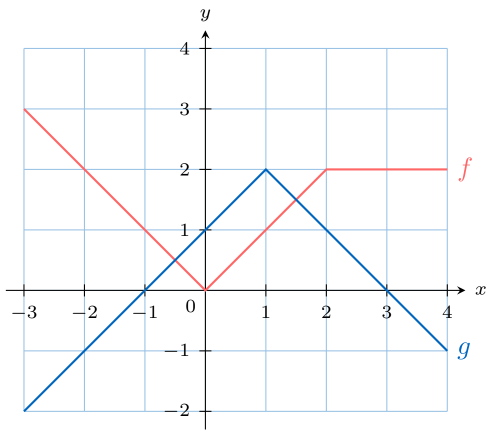
- $f(x) = \boxed{x^{-4}}$
- $g(x) = \boxed{x^3 - 2}$

Answer: One possible pair is $f(x) = x^{-4}$ and $g(x) = x^3 - 2$, since

$$\begin{aligned} f(g(x)) &= f(x^3 - 2) \\ &= (x^3 - 2)^{-4}. \end{aligned}$$

B.2.3 EVALUATING COMPOSITE FUNCTIONS FROM GRAPHS

Ex 112: The graphs of two functions, f and g, are shown below.



Use the graphs to find the values of:

1. $(f \circ g)(1) = \boxed{2}$
2. $(g \circ f)(-2) = \boxed{1}$
3. $(f \circ f)(0) = \boxed{0}$

Answer:

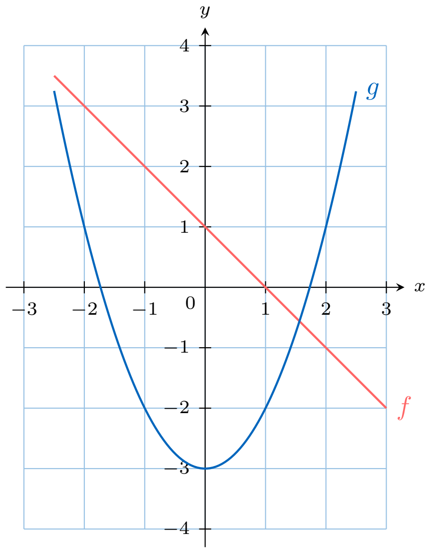
1. To find $(f \circ g)(1) = f(g(1))$:
   - First, find $g(1)$ from the graph of g. At $x = 1$, $g(1) = 2$.
   - Now, find $f(2)$ from the graph of f. At $x = 2$, $f(2) = 2$.

   So, $\boldsymbol{(f \circ g)(1) = 2}$.
2. To find $(g \circ f)(-2) = g(f(-2))$:
   - First, find $f(-2)$ from the graph of f. At $x = -2$, $f(-2) = 2$.
   - Now, find $g(2)$ from the graph of g. At $x = 2$, $g(2) = 1$.

   So, $\boldsymbol{(g \circ f)(-2) = 1}$.
3. To find $(f \circ f)(0) = f(f(0))$:
   - First, find $f(0)$ from the graph of f. At $x = 0$, $f(0) = 0$.
   - Now, find $f(0)$ again. $f(0) = 0$.

   So, $\boldsymbol{(f \circ f)(0) = 0}$.

Ex 113: The graphs of two functions, f and g, are shown.



Find the values of:

1. $(f \circ g)(2) = \boxed{0}$
2. $(g \circ f)(-1) = \boxed{1}$

Answer:

1. To find $(f \circ g)(2) = f(g(2))$:
   - First, from the graph of g, we locate $x = 2$ and read the corresponding y-value, which is $g(2) = 1$.
   - Now we find $f(1)$ from the graph of f. At $x = 1$, we read the y-value, which is $f(1) = 0$.

   So, $\boldsymbol{(f \circ g)(2) = 0}$.
2. To find $(g \circ f)(-1) = g(f(-1))$:
   - First, from the graph of f, we locate $x = -1$ and read the y-value, which is $f(-1) = 2$.
   - Now we find $g(2)$ from the graph of g. At $x = 2$, we read the y-value, which is $g(2) = 1$.

   So, $\boldsymbol{(g \circ f)(-1) = 1}$.

B.2.4 SOLVING EQUATIONS WITH COMPOSITE FUNCTIONS

Ex 114: Let $f(x) = x^2 - 3$ and $g(x) = 2x - 1$. Find all values of x such that $(f \circ g)(x) = 6$.

Answer: First, we find an expression for the composite function $(f \circ g)(x)$.

$$(f \circ g)(x) = f(g(x)) = f(2x - 1) = (2x - 1)^2 - 3$$

Now, we set this expression equal to 6 and solve for x.

$$\begin{aligned} (2x - 1)^2 - 3 &= 6 \\ (2x - 1)^2 &= 9 \\ \sqrt{(2x - 1)^2} &= \pm\sqrt{9} \\ 2x - 1 &= \pm 3 \end{aligned}$$

This gives two separate linear equations:

- $2x - 1 = 3 \Leftrightarrow 2x = 4 \Leftrightarrow x = 2$
- $2x - 1 = -3 \Leftrightarrow 2x = -2 \Leftrightarrow x = -1$

The solutions are $\boldsymbol{x=2}$ and $\boldsymbol{x=-1}$.

Ex 115: Let $f(x)=2x-5$ and $g(x)=\dfrac{x+1}{3}$. Find all values of x such that $(g\circ f)(x)=x$.

Answer: First, we find an expression for the composite function $(g\circ f)(x)$.

$$(g\circ f)(x)=g(f(x))=g(2x-5)=\frac{(2x-5)+1}{3}=\frac{2x-4}{3}$$

Now, we set this expression equal to x and solve.

$$\begin{aligned}\frac{2x-4}{3}&=x\\ 2x-4&=3x && \text{(multiply both sides by 3)}\\ -4&=3x-2x && \text{(subtract 2x from both sides)}\\ x&=-4\end{aligned}$$

The solution is $\boldsymbol{x=-4}$.

Ex 116: Let $f(x)=x^2-4x+5$ and $g(x)=x-1$. Find all values of x such that $(f\circ g)(x)=2$.

Answer: First, we find an expression for the composite function $(f\circ g)(x)$.

$$\begin{aligned}(f\circ g)(x)&=f(g(x))\\ &=f(x-1)\\ &=(x-1)^2-4(x-1)+5\\ &=(x^2-2x+1)-(4x-4)+5 && \text{(expand the terms)}\\ &=x^2-2x+1-4x+4+5\\ &=x^2-6x+10 && \text{(simplify)}\end{aligned}$$

Now, we set this expression equal to 2 and solve the resulting quadratic equation.

$$\begin{aligned}x^2-6x+10&=2\\ x^2-6x+8&=0 && \text{(subtract 2 from both sides)}\\ (x-2)(x-4)&=0 && \text{(factor the quadratic)}\end{aligned}$$

This gives two solutions:

$$x-2=0\Leftrightarrow x=2$$

$$x-4=0\Leftrightarrow x=4$$

The solutions are $\boldsymbol{x=2}$ and $\boldsymbol{x=4}$.

B.3 INVERSE FUNCTIONS

B.3.1 FINDING AND CHECKING INVERSES

Ex 117:

1. Find the inverse of $f(x)=x+3$.

$$f^{-1}(x)=\boxed{x-3}$$

2. Evaluate

$$f^{-1}(f(x))=\boxed{x}$$
$$f(f^{-1}(x))=\boxed{x}$$

Answer:

1. Set $y=x+3$.

$$\begin{aligned}y&=x+3\\ x&=y-3\end{aligned}$$

So, the inverse function is $f^{-1}(x)=x-3$.

2. $$\begin{aligned}f^{-1}(f(x))&=f^{-1}(x+3)\\ &=(x+3)-3\\ &=x\end{aligned}$$

3. $$\begin{aligned}f(f^{-1}(x))&=f(x-3)\\ &=(x-3)+3\\ &=x\end{aligned}$$

Ex 118:

1. Find the inverse of $f(x)=4x-8$.

$$f^{-1}(x)=\boxed{\frac{x+8}{4}}$$

2. Evaluate

$$f^{-1}(f(x))=\boxed{x}$$
$$f(f^{-1}(x))=\boxed{x}$$

Answer:

1. Set $y=4x-8$.

$$\begin{aligned}y&=4x-8\\ y+8&=4x\\ x&=\frac{y+8}{4}\end{aligned}$$

So, the inverse function is $f^{-1}(x)=\frac{x+8}{4}$.

2. $$\begin{aligned}f^{-1}(f(x))&=f^{-1}(4x-8)\\ &=\frac{(4x-8)+8}{4}\\ &=\frac{4x}{4}\\ &=x\end{aligned}$$

3. $$\begin{aligned}f(f^{-1}(x))&=f\left(\frac{x+8}{4}\right)\\ &=4\times\frac{x+8}{4}-8\\ &=(x+8)-8\\ &=x\end{aligned}$$

Ex 119:

1. Find the inverse of $f(x)=\dfrac{x}{2}-3$.

$$f^{-1}(x)=\boxed{2(x+3)}$$

2. Evaluate

$$f^{-1}(f(x))=\boxed{x}$$
$$f(f^{-1}(x))=\boxed{x}$$

Answer:

1. Set $y = \frac{x}{2} - 3$.

$$\begin{aligned} y &= \frac{x}{2} - 3 \\ y + 3 &= \frac{x}{2} \\ 2(y+3) &= x \\ x &= 2(y+3) \end{aligned}$$

So, the inverse function is $f^{-1}(x) = 2(x+3)$.

2.

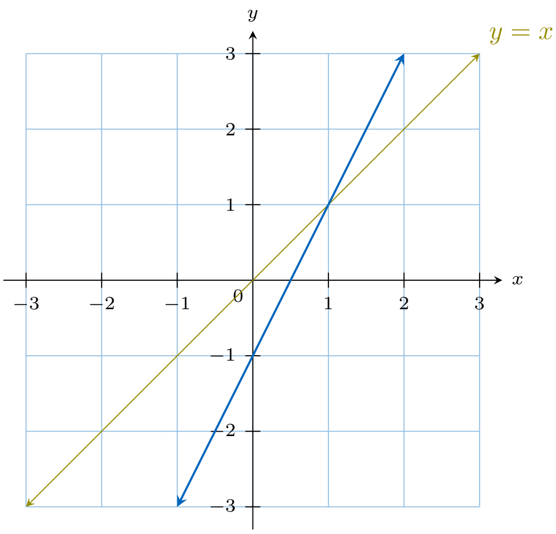
$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}\left(\frac{x}{2} - 3\right) \\ &= 2\left(\frac{x}{2} - 3 + 3\right) \\ &= 2 \times \frac{x}{2} \\ &= x \end{aligned}$$

3.

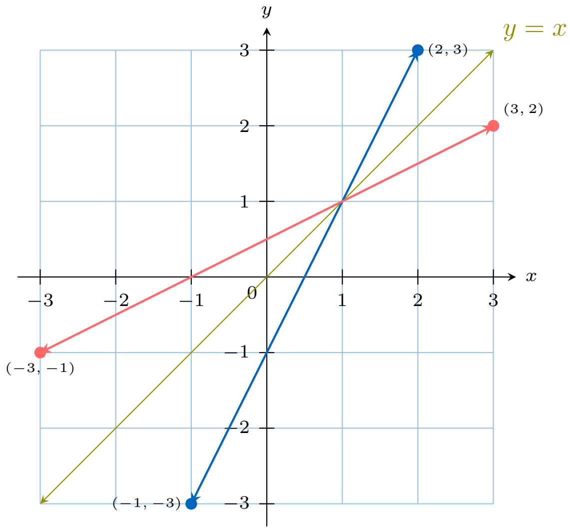
$$\begin{aligned} f(f^{-1}(x)) &= f\left(2(x+3)\right) \\ &= \frac{2(x+3)}{2} - 3 \\ &= (x+3) - 3 \\ &= x \end{aligned}$$

B.3.2 GRAPHING THE INVERSE FUNCTION BY REFLECTION

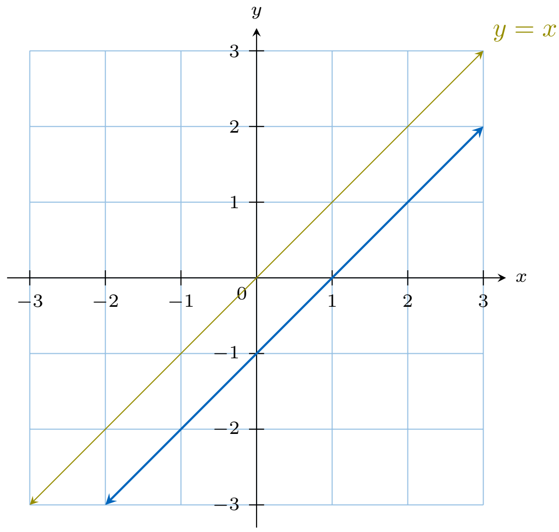
Ex 120: Draw the graph of the inverse function of the blue graph:



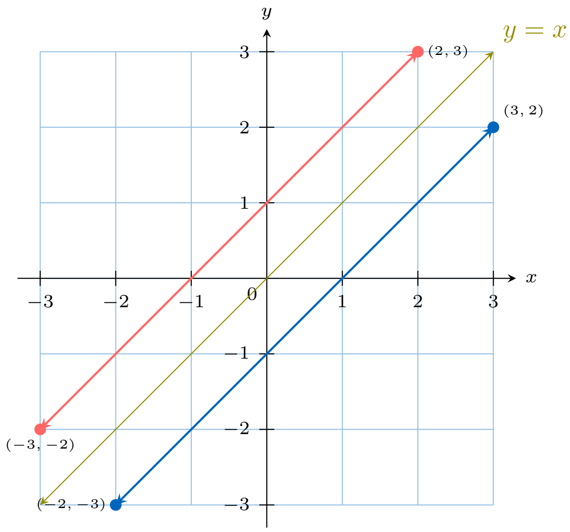
Answer: To draw the inverse, notice that the graph of f^{-1} is the reflection of the graph of f across the line $y = x$. You can plot two points on the blue line (for example, $(-1,-3)$ and $(2,3)$), then swap their coordinates to get points $(-3,-1)$ and $(3,2)$ on the inverse. Draw the line passing through these points: this is $y = \frac{x+1}{2}$, shown below in red.



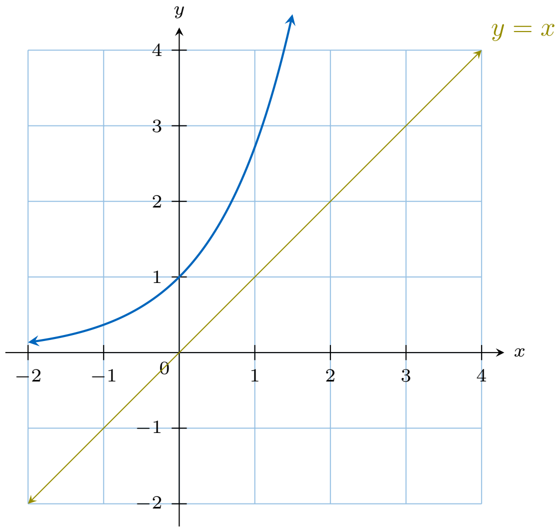
Ex 121: Draw the graph of the inverse function of the blue graph:



Answer: To draw the inverse, notice that the graph of f^{-1} is the reflection of the graph of f across the line $y = x$. For instance, the blue line contains the points $(-2,-3)$ and $(3,2)$. Swap their coordinates to get $(-3,-2)$ and $(2,3)$ on the inverse. Draw the line passing through these points: this is $y = x+1$, shown below in red.



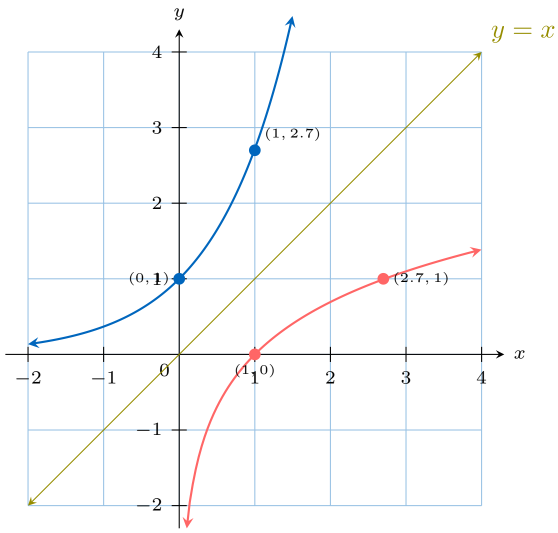
Ex 122: Draw the graph of the inverse function of the blue graph:



Answer: To draw the inverse graph, plot a few symmetric points:

- Take points from the blue curve, such as $(0, 1)$ and $(1, 2.7)$.
- Find their symmetric points with respect to the line $y = x$, i.e., swap their coordinates: $(1, 0)$ and $(2.7, 1)$.
- Plot these new points.
- Draw a smooth curve through the symmetric points; this is the graph of the inverse function.

You do **not** need to know the exact equation of the curve!



Remark: The inverse graph is obtained by reflecting each point of the blue curve across the line $y = x$.

B.3.3 FINDING INVERSES OF VARIOUS FUNCTION TYPES

Ex 123: Let the function f be defined by

$$\begin{aligned} f : [1, \infty) &\longrightarrow [2, \infty) \\ x &\longmapsto (x-1)^2 + 2 \end{aligned}$$

1. State the domain and range of f.
2. Find an expression for $f^{-1}(x)$.
3. State the domain and range of f^{-1}.

Answer:

1. The vertex of the parabola is at $(1, 2)$. Since the domain is $x \geq 1$, we are on the right arm of the parabola which opens upwards.
   - **Domain**: Given as $\mathbf{[1, \infty)}$.
   - **Range**: The minimum value is at the vertex, so the range is $\mathbf{[2, \infty)}$.
2. To find the inverse, we set $y = f(x)$ and solve for x.
$$\begin{aligned} y &= (x-1)^2 + 2 \\ y - 2 &= (x-1)^2 \\ \pm\sqrt{y-2} &= x - 1 \end{aligned}$$
   Since the domain of f is $x \geq 1$, the range of f^{-1} must be $y \geq 1$. To ensure this, we must choose the positive root.
$$\sqrt{y-2} = x - 1 \Leftrightarrow x = 1 + \sqrt{y-2}$$
   Swapping variables gives $\boldsymbol{f^{-1}(x) = 1 + \sqrt{x-2}}$.
3. The domain and range of the inverse are swapped from the original function.
   - **Domain of $\boldsymbol{f^{-1}}$**: This is the range of f, which is $\mathbf{[2, \infty)}$.
   - **Range of $\boldsymbol{f^{-1}}$**: This is the domain of f, which is $\mathbf{[1, \infty)}$.

Ex 124: Let the function f be defined by $f(x) = \dfrac{2x+1}{x-3}$ for $x \neq 3$.

1. Find an expression for $f^{-1}(x)$.
2. Solve the equation $f^{-1}(x) = 4$.

Answer:

1. We set $y = f(x)$ and solve for x.
$$\begin{aligned} y &= \frac{2x+1}{x-3} \\ y(x-3) &= 2x+1 && \text{(multiply by } x-3\text{)} \\ yx - 3y &= 2x + 1 \\ yx - 2x &= 3y + 1 && \text{(collect x-terms)} \\ x(y-2) &= 3y+1 && \text{(factor out x)} \\ x &= \frac{3y+1}{y-2} \end{aligned}$$
   Swapping the variables gives $\boldsymbol{f^{-1}(x) = \frac{3x+1}{x-2}}$.
2. We now solve the equation $f^{-1}(x) = 4$.
$$\begin{aligned} \frac{3x+1}{x-2} &= 4 \\ 3x + 1 &= 4(x-2) \\ 3x + 1 &= 4x - 8 \\ 9 &= x \end{aligned}$$
   The solution is $\boldsymbol{x = 9}$.

Ex 125: Consider the function $f(x) = \dfrac{5}{x}$ for $x \neq 0$.

1. Find the inverse function, $f^{-1}(x)$.

2. Compare $f(x)$ and $f^{-1}(x)$ and explain what it means for a function to be its own inverse.

Answer:

1. We set $y = f(x)$ and solve for x.

$$\begin{aligned} y &= \frac{5}{x} \\ xy &= 5 \\ x &= \frac{5}{y} \end{aligned}$$

   Swapping the variables gives $\boldsymbol{f^{-1}(x) = \frac{5}{x}}$.

2. By comparing the expressions, we see that $\boldsymbol{f(x) = f^{-1}(x)}$. A function that is its own inverse is called a **self-inverse function** or an **involution**. This means that applying the function twice returns the original input: $(f \circ f)(x) = x$. Graphically, this implies that the graph of the function is perfectly symmetric about the line $y = x$.