FUNCTION TRANSFORMATIONS

Function transformations allow us to take a basic "parent" function and modify it to create a new, related function. By applying a sequence of transformations, we can shift, stretch, compress, or reflect the graph of the parent function. Understanding these transformations is essential, as it allows us to predict the graph of a complex function based on a simpler one and to model real-world phenomena by adjusting a basic function to fit observed data. In this chapter, we will explore vertical and horizontal translations, dilations (stretches/compressions), and reflections.

A TRANSLATION

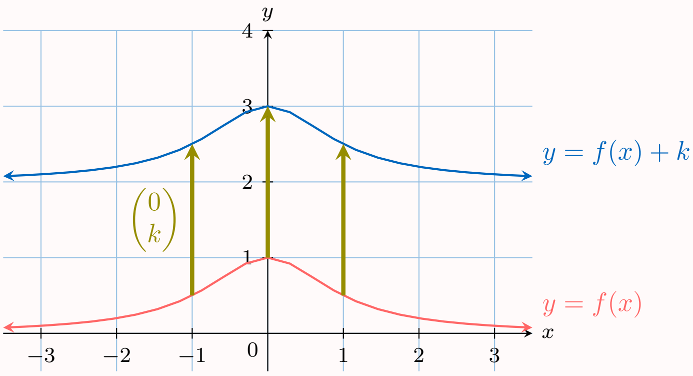
Definition Vertical Translation by Adding a Constant

A **vertical translation** shifts the graph of a function up or down. The transformation is defined by:

$$g(x) = f(x) + k$$

This transformation maps a point (x, y) on the graph of f to a new point $(x, y + k)$ on the graph of g.

- If $k > 0$, the graph is translated $\boldsymbol{k}$ units **up**.
- If $k < 0$, the graph is translated $|\boldsymbol{k}|$ units **down**.



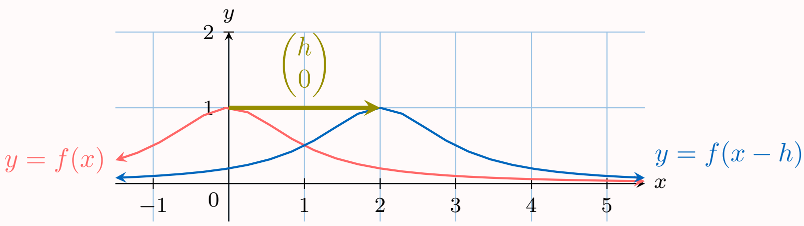
Definition Horizontal Translation

A **horizontal translation** shifts the graph of a function left or right. The transformation is defined by:

$$g(x) = f(x - h)$$

This transformation maps a point (x, y) on the graph of f to a new point $(x + h, y)$ on the graph of g.

- If $h > 0$, the graph is translated $\boldsymbol{h}$ units to the **right**.
- If $h < 0$, the graph is translated $|\boldsymbol{h}|$ units to the **left**.



B DILATION

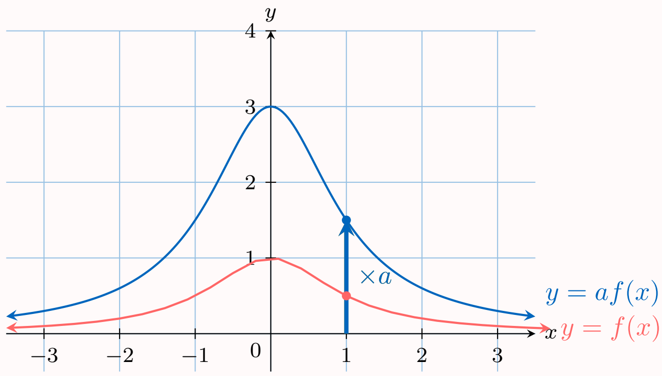
Definition Vertical Dilation

A **vertical dilation** stretches or compresses the graph of a function vertically. It is defined by:

$$g(x) = a \cdot f(x)$$

This transformation maps a point (x, y) to (x, ay).

- If $|a| > 1$, the graph is **stretched** vertically by a factor of a.
- If $0 < |a| < 1$, the graph is **compressed** vertically by a factor of a.



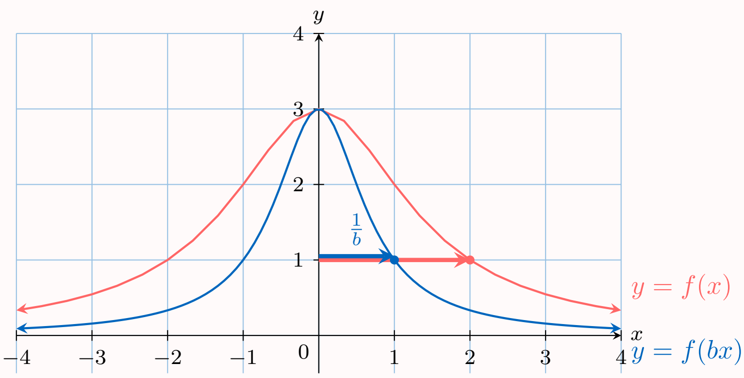
Definition Horizontal Dilation

A **horizontal dilation** stretches or compresses the graph of a function horizontally. It is defined by:

$$g(x) = f(bx)$$

This transformation maps a point (x, y) to a new point $(\frac{x}{b}, y)$.

- If $|b| > 1$, the graph is **compressed** horizontally by a factor of $\frac{1}{b}$.
- If $0 < |b| < 1$, the graph is **stretched** horizontally by a factor of $\frac{1}{b}$.



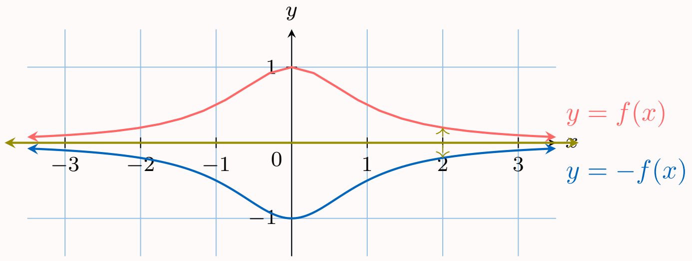
C REFLECTION

Definition Reflection in the x-axis

A **reflection in the x-axis** flips the graph of a function vertically. It is a special case of vertical dilation where $a = -1$. It is defined by:

$$g(x) = -f(x)$$

This transformation maps a point (x, y) to $(x, -y)$.

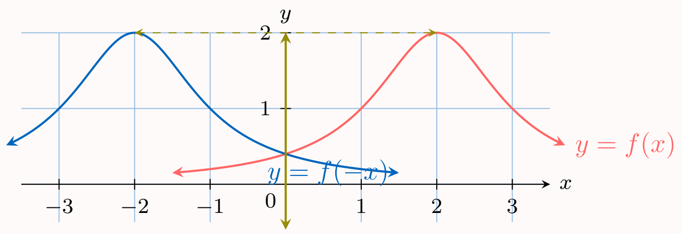


Definition Reflection in the y-axis

A **reflection in the y-axis** flips the graph of a function horizontally. It is a special case of horizontal dilation where $b = -1$. It is defined by:

$$g(x) = f(-x)$$

This transformation maps a point (x, y) to $(-x, y)$.



D COMBINING TRANSFORMATIONS

When multiple transformations are applied to a function, the order in which they are performed is crucial. The standard form for a transformed function is:

$$g(x) = a \cdot f(b(x - c)) + d$$

To graph this function from the parent function $f(x)$, apply the transformations in the following order:

Method Order of Transformations

1. **Horizontal Transformations (inside the brackets):**
   - Apply the horizontal stretch/compression by the factor $\frac{1}{b}$.
   - Apply the horizontal translation (shift) by c units.
2. **Vertical Transformations (outside the brackets):**
   - Apply the vertical stretch/compression by the factor a.
   - Apply the vertical translation (shift) by d units.

Note: Always factor out the coefficient b from the term inside the function to correctly identify the horizontal shift c. For example, transform $f(2x - 6)$ as $f(2(x - 3))$. This shows a compression by $\frac{1}{2}$ followed by a shift of 3 units right, not 6.

Ex: Describe the sequence of transformations that maps the graph of $f(x) = \sqrt{x}$ onto the graph of $g(x) = 3\sqrt{-x+2} - 4$.

Answer: First, rewrite the function $g(x)$ in the standard form $a \cdot f(b(x - c)) + d$:

$$g(x) = 3\sqrt{-(x-2)} - 4$$

Comparing this to $f(x) = \sqrt{x}$, we have $a = 3$, $b = -1$, $c = 2$, and $d = -4$.The transformations are applied as follows:

1. **Horizontal Transformations on $f(x) = \sqrt{x}$:**

- **Reflection:** Since $b = -1$, reflect the graph across the y-axis ($y = \sqrt{-x}$).
- **Translation:** Since $c = 2$, translate the graph 2 units to the right ($y = \sqrt{-(x-2)}$).

2. **Vertical Transformations on the result:**

- **Stretch:** Since $a = 3$, stretch the graph vertically by a factor of 3 ($y = 3\sqrt{-(x-2)}$).
- **Translation:** Since $d = -4$, translate the graph 4 units down ($y = 3\sqrt{-(x-2)} - 4$).