FUNCTION TRANSFORMATIONS

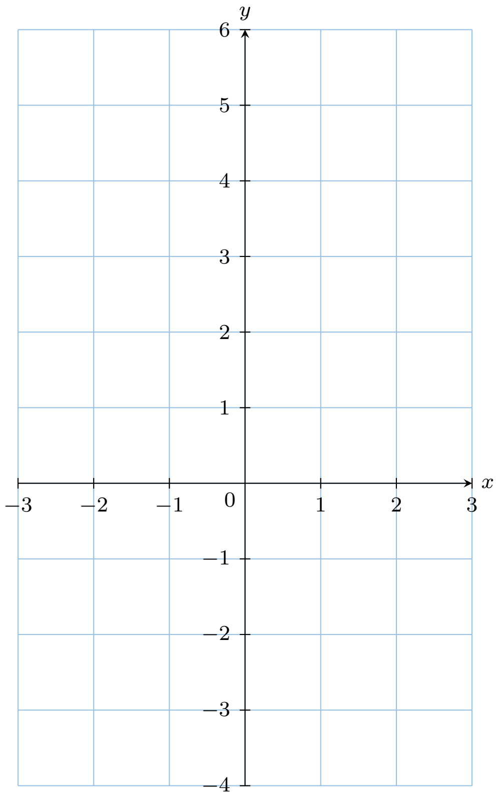
A TRANSLATION

A.1 TRANSLATING GRAPHS VERTICALLY

Ex 1: For the functions $f(x) = x$ and $g(x) = x + 3$:

1. On the same set of axes, sketch the graphs of f and g. (You may fill in a table of values for $x = -3, -2, -1, 0, 1, 2, 3$.)

2. Find the geometrical transformation between these two graphs.

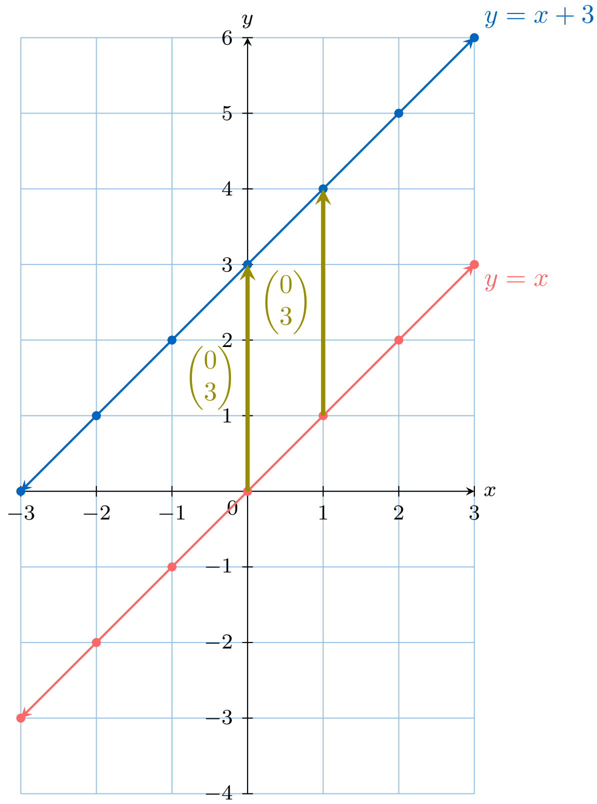


Answer:

1. Fill in the table of values:

x	-3	-2	-1	0	1	2	3
$f(x)$	-3	-2	-1	0	1	2	3
$g(x)$	0	1	2	3	4	5	6

Plot the points and draw both lines:

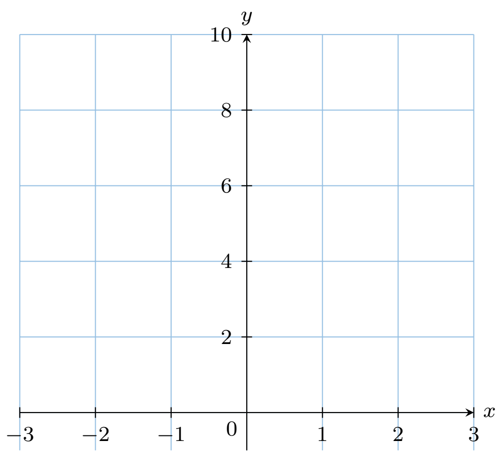


2. The graph of g is obtained by translating the graph of f by the vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$. This is a **vertical translation** upward by 3 units, because $g(x) = f(x) + 3$.

Ex 2: For the functions $f(x) = x^2$ and $g(x) = x^2 + 2$:

1. On the same set of axes, sketch the graphs of f and g. (You may fill in a table of values for $x = -3, -2, -1, 0, 1, 2, 3$.)

2. Find the geometrical transformation between these two graphs.

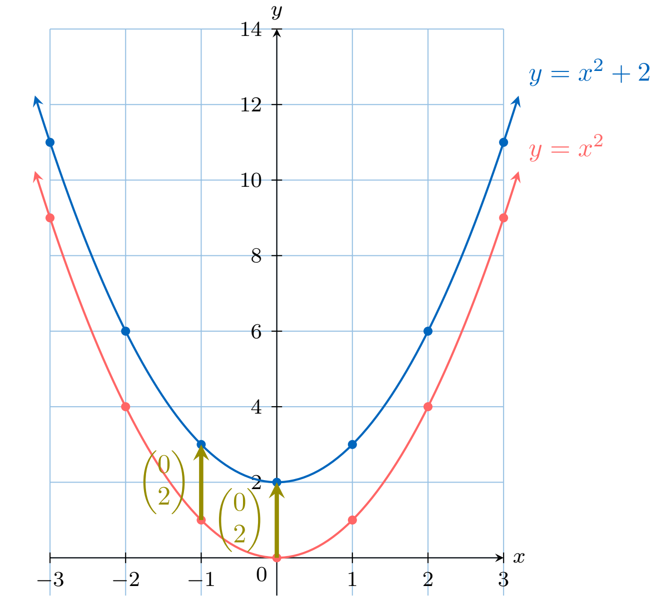


Answer:

1. Fill in the table of values:

x	-3	-2	-1	0	1	2	3
$f(x)$	9	4	1	0	1	4	9
$g(x)$	11	6	3	2	3	6	11

Plot the points and draw the two parabolas:

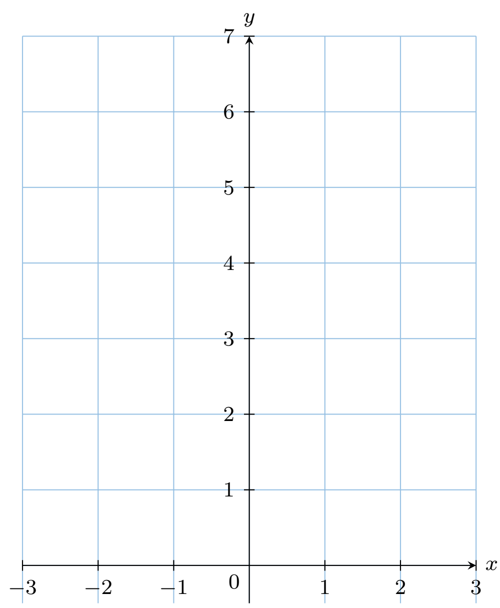


2. The graph of g is obtained by translating the graph of f by the vector $\begin{pmatrix}0\\2\end{pmatrix}$. This is a **vertical translation** upward by 2 units, because $g(x) = f(x) + 2$.

Ex 3: For the functions $f(x) = \dfrac{4}{1+x^2}$ and $g(x) = \dfrac{4}{1+x^2} + 3$:

1. On the same set of axes, sketch the graphs of f and g. (You may fill in a table of values for $x = -3, -2, -1, 0, 1, 2, 3$.)

2. Find the geometrical transformation between these two graphs.

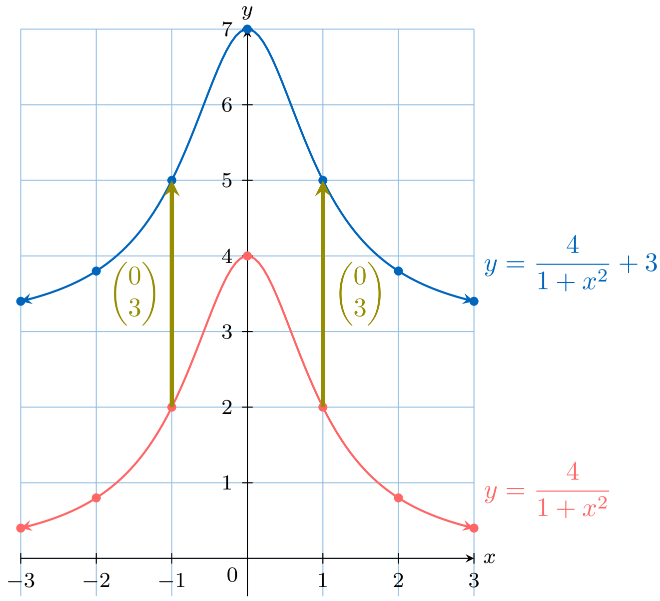


Answer:

1. Fill in the table of values:

x	-3	-2	-1	0	1	2	3
$f(x)$	0.4	0.8	2	4	2	0.8	0.4
$g(x)$	3.4	3.8	5	7	5	3.8	3.4

Plot the points and draw both curves :

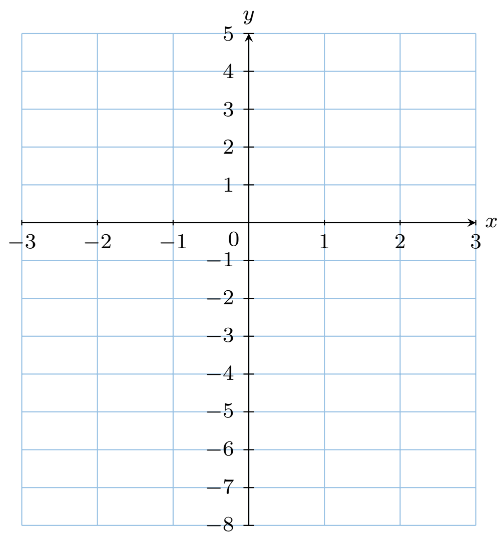


2. The graph of g is obtained by translating the graph of f by the vector $\begin{pmatrix}0\\3\end{pmatrix}$. This is a **vertical translation** upward by 3 units, because $g(x) = f(x) + 3$.

Ex 4: For the functions $f(x) = -(x-2)(x+2)$ and $g(x) = -(x-2)(x+2) - 2$:

1. On the same set of axes, sketch the graphs of f and g. (You may fill in a table of values for $x = -3, -2, -1, 0, 1, 2, 3$.)

2. Find the geometrical transformation between these two graphs.

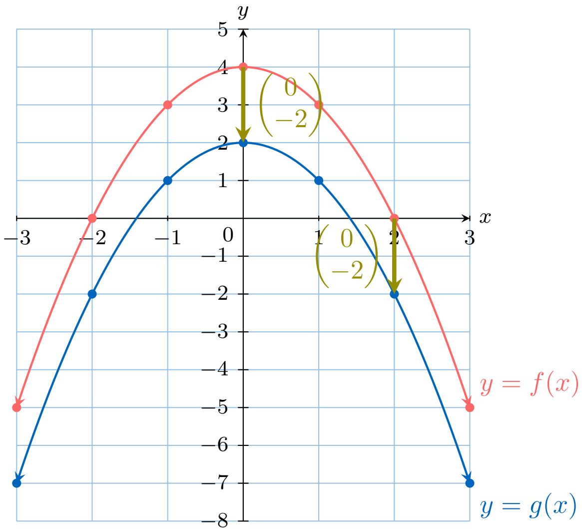


Answer:

1. Fill in the table of values:

x	-3	-2	-1	0	1	2	3
$f(x)$	-5	0	3	4	3	0	-5
$g(x)$	-7	-2	1	2	1	-2	-7

Plot the points and draw both parabolas :



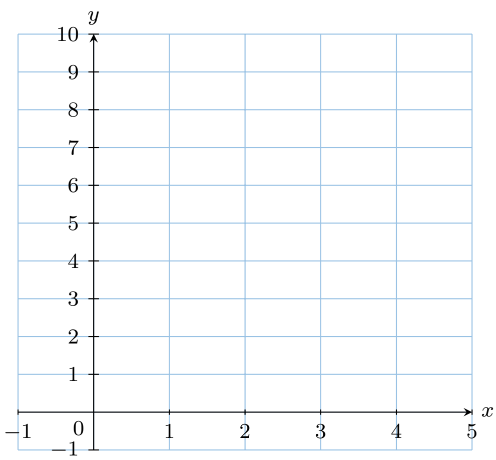
2. The graph of g is obtained by translating the graph of f by the vector $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$. This is a **vertical translation** downward by 2 units, car $g(x) = f(x) - 2$.

A.2 TRANSLATING GRAPHS HORIZONTALLY

Ex 5: For the functions $f(x) = x^2$ and $g(x) = (x-3)^2$:

1. On the same set of axes, sketch the graphs of f and g. (You may fill in a table of values for $x = -1, 0, 1, 2, 3, 4, 5$.)

2. Find the geometrical transformation between these two graphs.

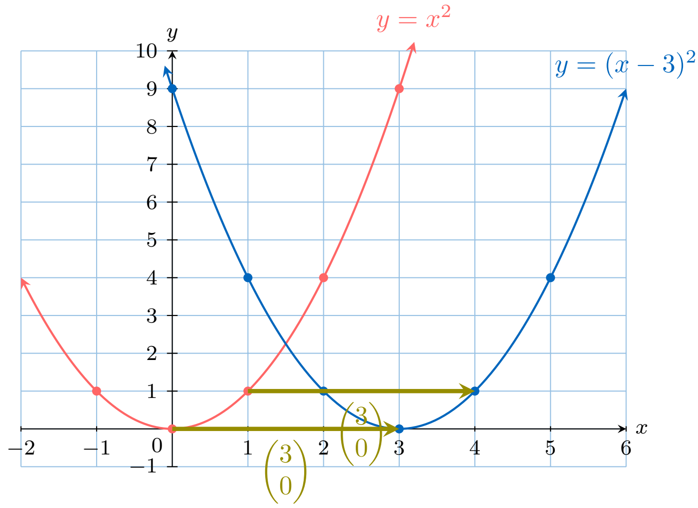


Answer:

1. Fill in the table of values:

x	-1	0	1	2	3	4	5
$f(x)$	1	0	1	4	9	16	25
$g(x)$	16	9	4	1	0	1	4

Plot the points and draw both curves:

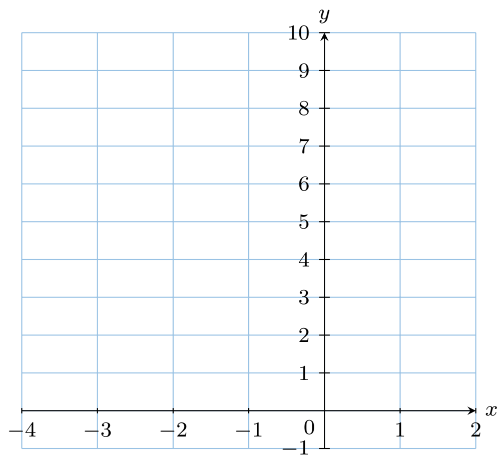


2. The graph of g is obtained by translating the graph of f by the vector $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$. This is a **horizontal translation** to the right by 3 units, because $g(x) = f(x-3)$.

Ex 6: For the functions $f(x) = x^2$ and $g(x) = (x+2)^2$:

1. On the same set of axes, sketch the graphs of f and g. (You may fill in a table of values for $x = -4, -3, -2, -1, 0, 1, 2$.)

2. Find the geometrical transformation between these two graphs.

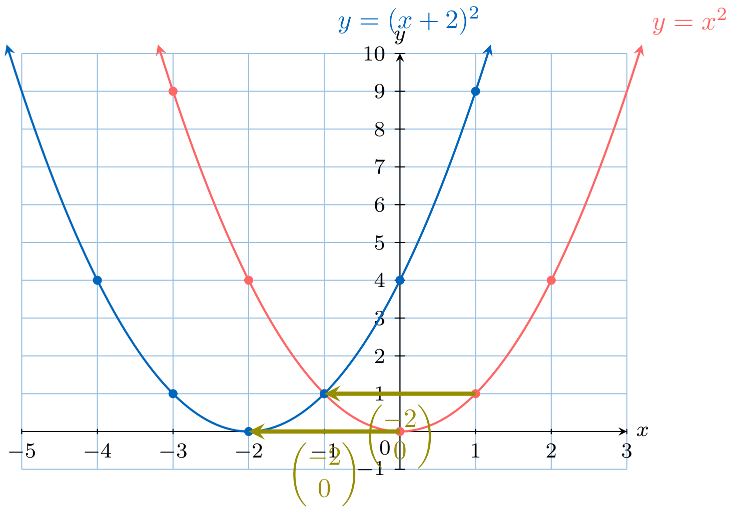


Answer:

1. Fill in the table of values:

x	-4	-3	-2	-1	0	1	2
$f(x)$	16	9	4	1	0	1	4
$g(x)$	4	1	0	1	4	9	16

Plot the points and draw the two parabolas:

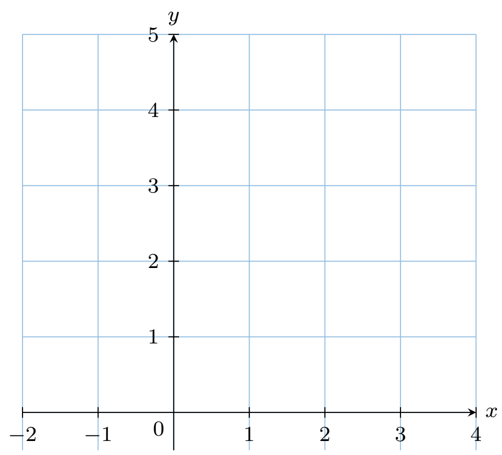


2. The graph of g is obtained by translating the graph of f by the vector $\begin{pmatrix}-2\\0\end{pmatrix}$. This is a **horizontal translation** to the left by 2 units, because $g(x) = f(x+2)$.

Ex 7: For the functions $f(x) = \dfrac{4}{1+x^2}$ and $g(x) = \dfrac{4}{1+(x-2)^2}$:

1. On the same set of axes, sketch the graphs of f and g. (You may fill in a table of values for $x = -2, -1, 0, 1, 2, 3, 4$.)

2. Find the geometrical transformation between these two graphs.

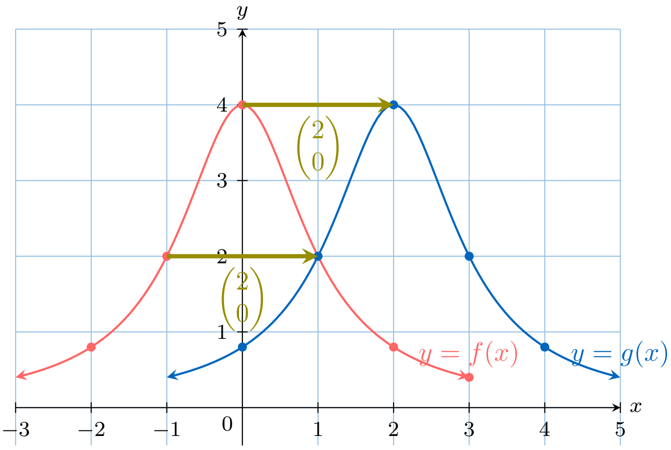


Answer:

1. Fill in the table of values:

x	-2	-1	0	1	2	3	4
$f(x)$	0.8	2	4	2	0.8	0.4	0.24
$g(x)$	0.24	0.4	0.8	2	4	2	0.8

Plot the points and draw both curves :



2. The graph of g is obtained by translating the graph of f by the vector $\begin{pmatrix}2\\0\end{pmatrix}$. This is a **horizontal translation** to the right by 2 units, because $g(x) = f(x-2)$.

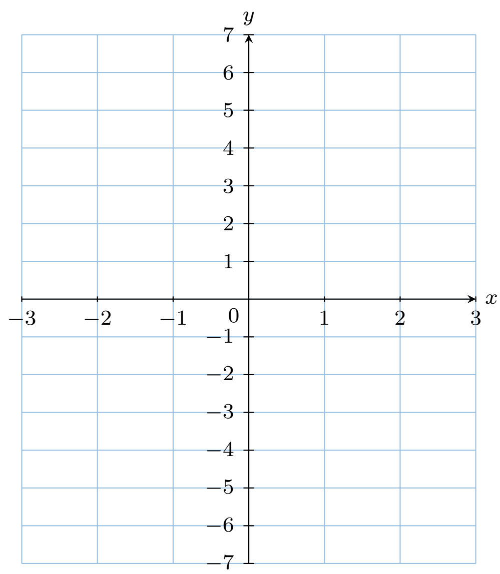
B DILATION

B.1 DILATING GRAPHS VERTICALLY

Ex 8: For the functions $f(x) = x$ and $g(x) = 2x$:

1. On the same set of axes, sketch the graphs of f and g. (You may fill in a table of values for $x = -3, -2, -1, 0, 1, 2, 3$.)

2. Find the geometrical transformation between these two graphs.

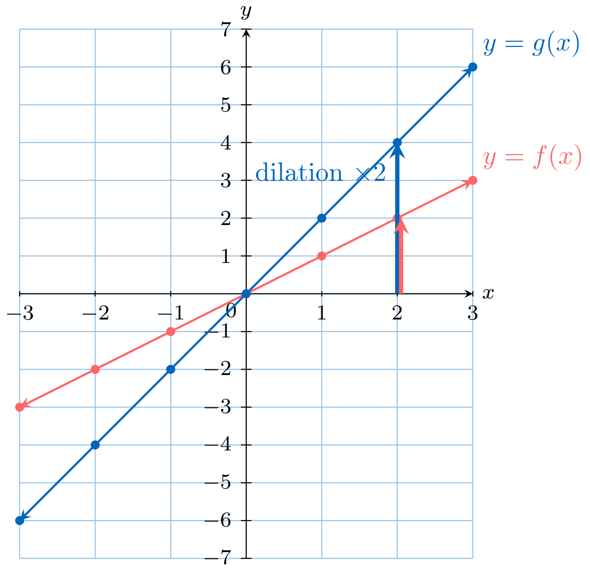


Answer:

1. Fill in the table of values:

x	-3	-2	-1	0	1	2	3
$f(x)$	-3	-2	-1	0	1	2	3
$g(x)$	-6	-4	-2	0	2	4	6

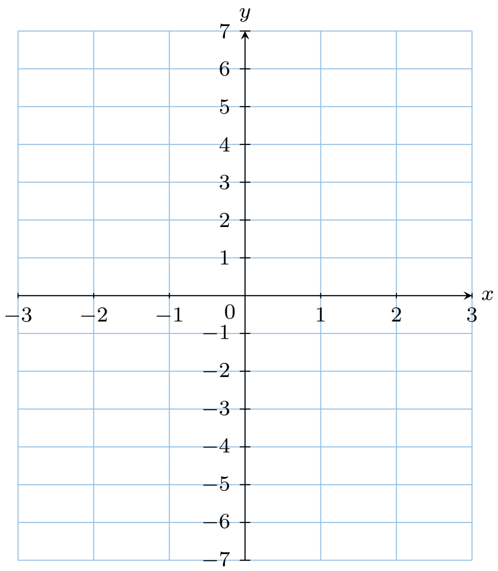
Plot the points and draw both lines :



2. The graph of g is obtained by vertically stretching (dilating) the graph of f by a factor of 2, because $g(x) = 2f(x)$.

Ex 9: For the functions $f(x) = \dfrac{x}{2}$ and $g(x) = 2x$:

1. On the same set of axes, sketch the graphs of f and g. (You may fill in a table of values for $x = -3, -2, -1, 0, 1, 2, 3$.)
2. Find the geometrical transformation between these two graphs.

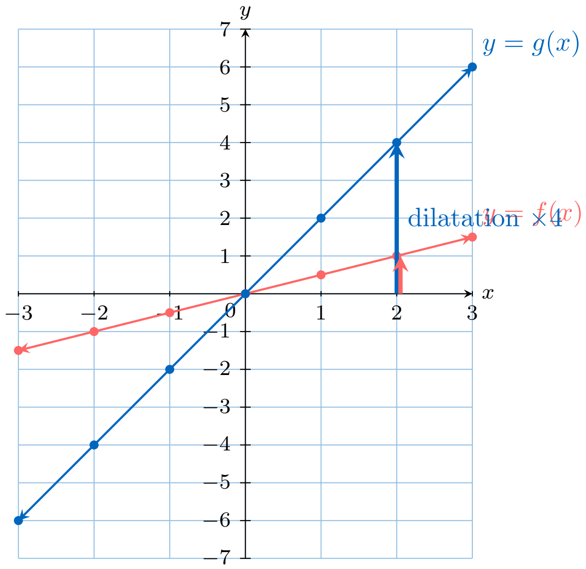


Answer:

1. Fill in the table of values:

x	-3	-2	-1	0	1	2	3
$f(x)$	-1.5	-1	-0.5	0	0.5	1	1.5
$g(x)$	-6	-4	-2	0	2	4	6

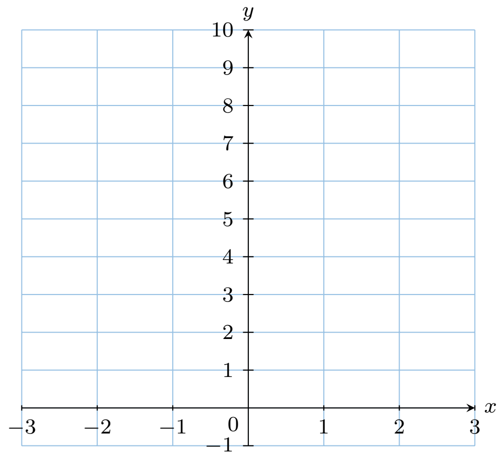
Plot the points and draw both lines :



2. The graph of g is obtained by vertically stretching (dilating) the graph of f by a factor of 4, because $g(x) = 4f(x)$.

Ex 10: For the functions $f(x) = x^2$ and $g(x) = \dfrac{x^2}{2}$:

1. On the same set of axes, sketch the graphs of f and g. (You may fill in a table of values for $x = -3, -2, -1, 0, 1, 2, 3$.)
2. Find the geometrical transformation between these two graphs.

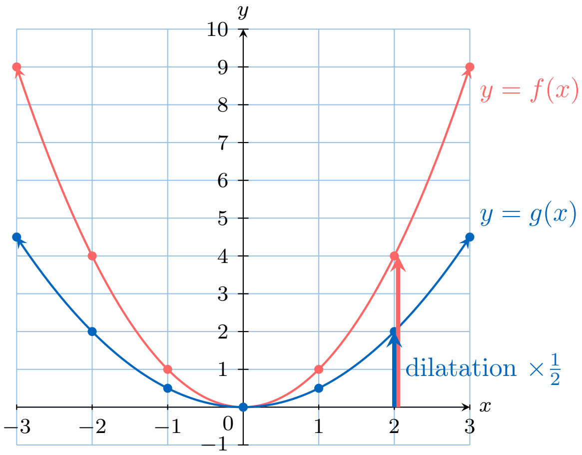


Answer:

1. Fill in the table of values:

x	-3	-2	-1	0	1	2	3
$f(x)$	9	4	1	0	1	4	9
$g(x)$	4.5	2	0.5	0	0.5	2	4.5

Plot the points and draw both parabolas :



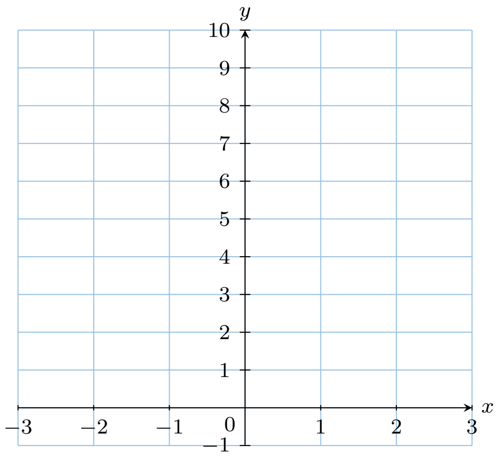
2. The graph of g is obtained by vertically compressing (dilating) the graph of f by a factor of $\frac{1}{2}$, because $g(x) = \frac{1}{2}f(x)$.

B.2 DILATING GRAPHS HORIZONTALLY

Ex 11: For the functions $f(x) = x^2$ and $g(x) = (2x)^2$:

1. On the same set of axes, sketch the graphs of f and g.

2. Find the geometrical transformation that maps the graph of f to the graph of g.

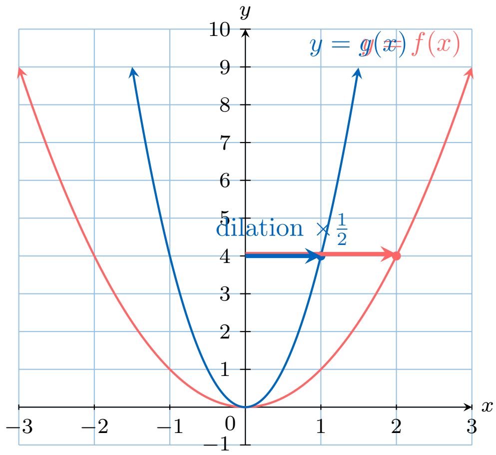


Answer:

1. Fill in a table of values:

x	-2	-1	0	1	2
$f(x) = x^2$	4	1	0	1	4
$g(x) = (2x)^2$	16	4	0	4	16

Plot the points and draw both curves:

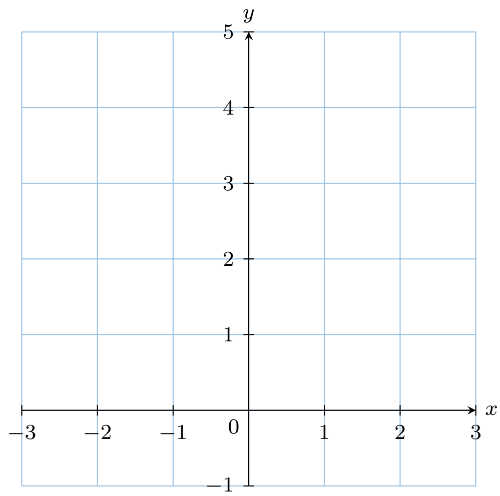


2. The graph of g is obtained by horizontally compressing the graph of f by a factor of $\frac{1}{2}$, because $g(x) = f(2x)$. A point (x, y) on the graph of f is mapped to $(\frac{x}{2}, y)$ on the graph of g. For example, the point $(2, 4)$ on f moves to $(1, 4)$ on g.

Ex 12: For the functions $f(x) = x^2$ and $g(x) = (\frac{1}{2}x)^2$:

1. On the same set of axes, sketch the graphs of f and g.

2. Find the geometrical transformation that maps the graph of f to the graph of g.

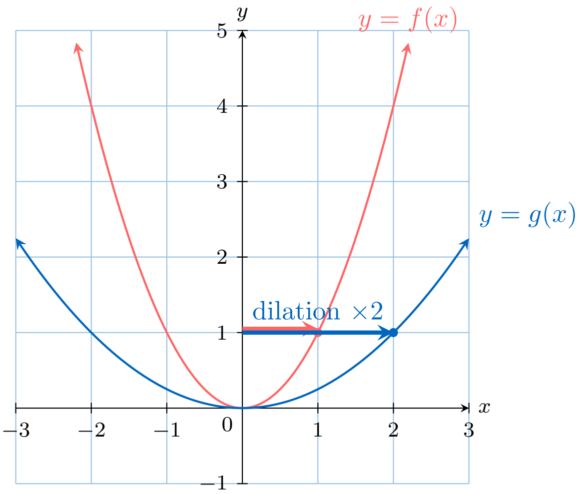


Answer:

1. Fill in a table of values:

x	-2	-1	0	1	2
$f(x) = x^2$	4	1	0	1	4
$g(x) = (\frac{1}{2}x)^2$	1	0.25	0	0.25	1

Plot the points and draw both curves:



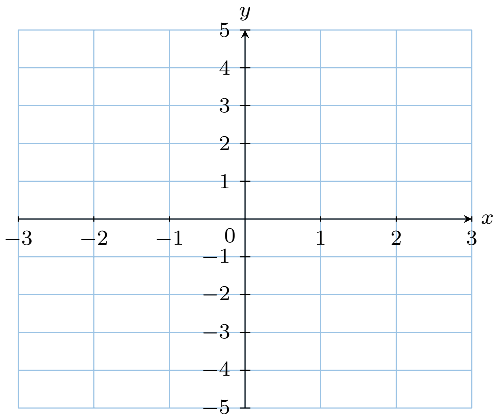
2. The graph of g is obtained by horizontally stretching the graph of f by a factor of 2, because $g(x) = f(\frac{1}{2}x)$. A point (x, y) on the graph of f is mapped to $(2x, y)$ on the graph of g. For example, the point $(1, 1)$ on f moves to $(2, 1)$ on g.

C REFLECTION

C.1 REFLECTING GRAPHS

Ex 13: For the functions $f(x) = x - 1$ and $g(x) = -(x - 1)$:

1. On the same set of axes, sketch the graphs of f and g. (You may fill in a table of values for $x = -3, -2, -1, 0, 1, 2, 3$.)
2. Find the geometrical transformation between these two graphs.

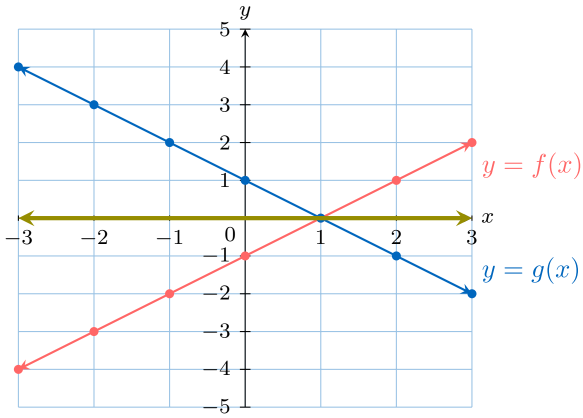


Answer:

1. Fill in the table of values:

x	-3	-2	-1	0	1	2	3
$f(x)$	-4	-3	-2	-1	0	1	2
$g(x)$	4	3	2	1	0	-1	-2

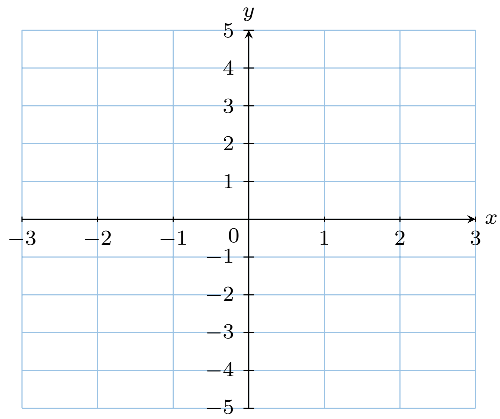
Plot the points and draw both lines :



2. The graph of g is obtained by reflecting the graph of f over the x-axis (symmetry about the x-axis) because $g(x) = -f(x)$.

Ex 14: For the functions $f(x) = x - 1$ and $g(x) = -x - 1$:

1. On the same set of axes, sketch the graphs of f and g. (You may fill in a table of values for $x = -3, -2, -1, 0, 1, 2, 3$.)
2. Find the geometrical transformation between these two graphs.

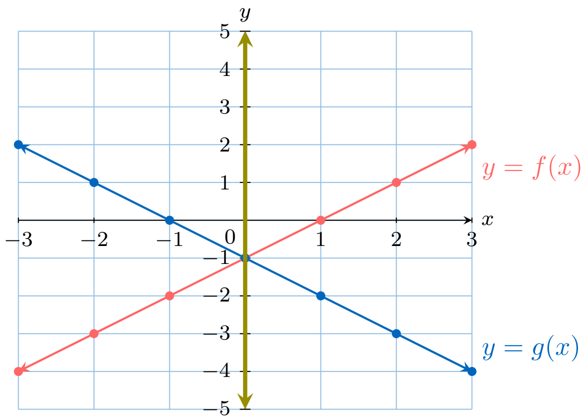


Answer:

1. Fill in the table of values:

x	-3	-2	-1	0	1	2	3
$f(x)$	-4	-3	-2	-1	0	1	2
$g(x)$	2	1	0	-1	-2	-3	-4

Plot the points and draw both lines :

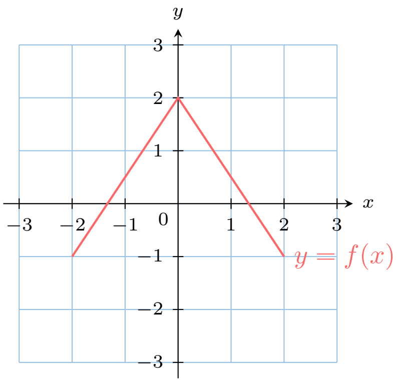


2. The graph of g is obtained by reflecting the graph of f over the y-axis (symmetry about the y-axis), because $g(x) = f(-x)$.

D COMBINING TRANSFORMATIONS

D.1 APPLYING COMBINED TRANSFORMATIONS

Ex 15: The graph of $y = f(x)$ is shown. On the same axes, sketch the graph of $y = f(x+1) - 2$.



Answer: The transformation $y = f(x+1) - 2$ involves two steps:

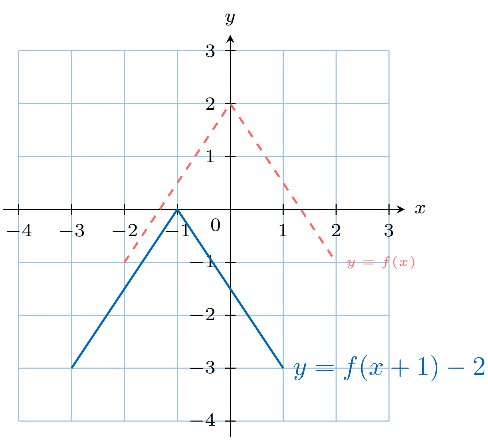
1. A horizontal translation of 1 unit to the left $(x+1)$.

2. A vertical translation of 2 units down (-2).

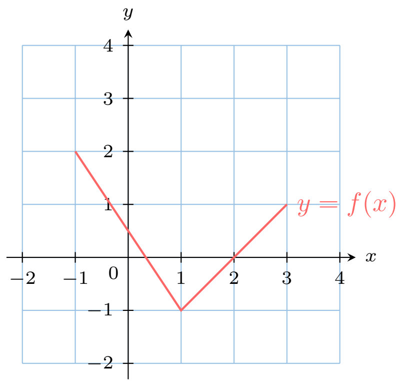
We apply these transformations to the key points of the original graph:

- $(-2,-1) \xrightarrow{\text{left 1, down 2}} (-3,-3)$
- $(0,2) \xrightarrow{\text{left 1, down 2}} (-1,0)$
- $(2,-1) \xrightarrow{\text{left 1, down 2}} (1,-3)$

Plotting the new points gives the transformed graph.



Ex 16: The graph of $y = f(x)$ is shown. On the same axes, sketch the graph of $y = 2f(x-1)$.



Answer: The transformation $y = 2f(x-1)$ involves two steps, applied in order:

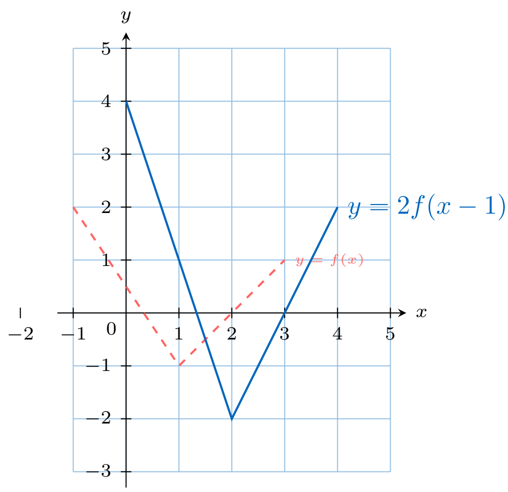
1. A horizontal translation of 1 unit to the right $(x-1)$.

2. A vertical stretch by a factor of 2 $(2f(\dots))$.

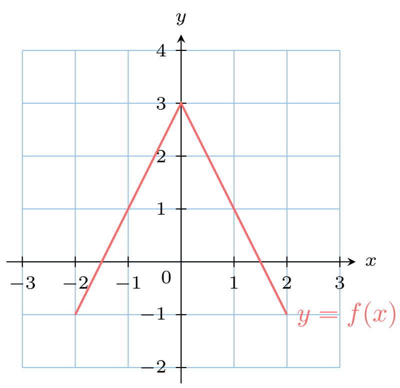
We apply these transformations to the key points:

- $(-1,2) \xrightarrow{\text{right 1}} (0,2) \xrightarrow{\text{stretch vert by 2}} (0,4)$
- $(1,-1) \xrightarrow{\text{right 1}} (2,-1) \xrightarrow{\text{stretch vert by 2}} (2,-2)$
- $(3,1) \xrightarrow{\text{righ 1}} (4,1) \xrightarrow{\text{stretch vert by 2}} (4,2)$

Plotting the new points gives the transformed graph.



Ex 17: The graph of $y = f(x)$ is shown. On the same axes, sketch the graph of $y = -f(x) + 1$.



Answer: The transformation $y = -f(x) + 1$ involves two vertical transformations:

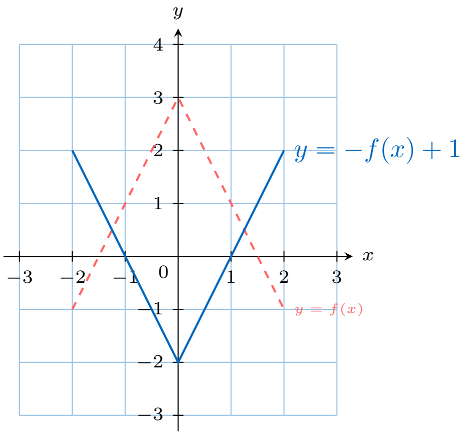
1. A reflection in the x-axis $(-f(x))$.

2. A vertical translation of 1 unit up $(+1)$.

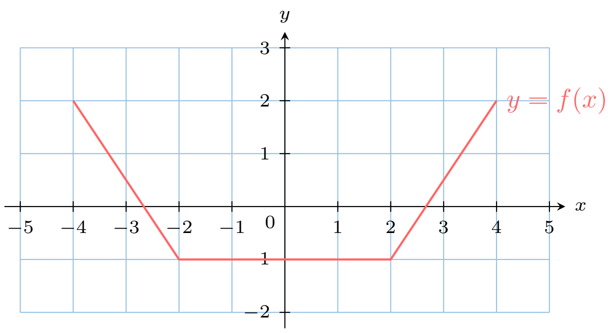
We apply these transformations to the key points of the original graph:

- $(-2,-1) \xrightarrow{\text{reflect in x-axis}} (-2,1) \xrightarrow{\text{up 1}} (-2,2)$
- $(0,3) \xrightarrow{\text{reflect in x-axis}} (0,-3) \xrightarrow{\text{up 1}} (0,-2)$
- $(2,-1) \xrightarrow{\text{reflect in x-axis}} (2,1) \xrightarrow{\text{up 1}} (2,2)$

Plotting the new points gives the transformed graph.



Ex 18: The graph of $y = f(x)$ is shown. On the same axes, sketch the graph of $y = f(2x) - 1$.



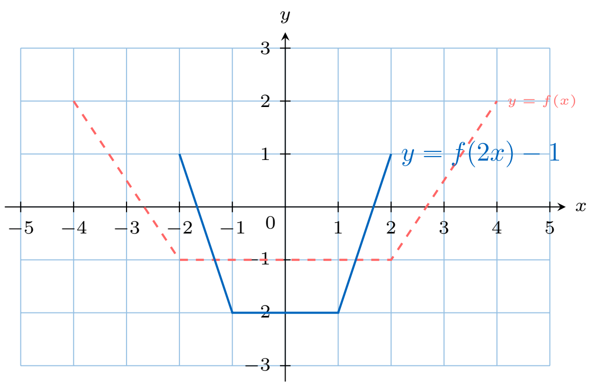
Answer: The transformation $y = f(2x) - 1$ involves two steps:

1. A horizontal dilation by a factor of $\frac{1}{2}$ $(f(2x))$.
2. A vertical translation of 1 unit down (-1).

We apply these transformations to the key points:

- $(-4,2) \xrightarrow{\text{horiz dilation by 1/2}} (-2,2) \xrightarrow{\text{down 1}} (-2,1)$
- $(-2,-1) \xrightarrow{\text{horiz dilation by 1/2}} (-1,-1) \xrightarrow{\text{down 1}} (-1,-2)$
- $(2,-1) \xrightarrow{\text{horiz dilation by 1/2}} (1,-1) \xrightarrow{\text{down 1}} (1,-2)$
- $(4,2) \xrightarrow{\text{horiz dilation by 1/2}} (2,2) \xrightarrow{\text{down 1}} (2,1)$

Plotting the new points gives the transformed graph.



D.2 FINDING EQUATIONS FROM A SEQUENCE OF TRANSFORMATIONS

Ex 19: Consider a function $y = f(x)$. Find the equation of the resulting function, $g(x)$, if the graph of f is transformed by the following sequence:

1. A reflection in the y-axis.
2. A horizontal stretch by a factor of 2.

Answer: We apply the transformations step-by-step to the initial function $y = f(x)$.

1. **Reflection in the y-axis**:
   This transformation replaces x with $-x$. The function becomes: $y = f(-x)$.
2. **Horizontal stretch by a factor of 2**:
   This transformation replaces x with $\frac{x}{2}$ in the current function. We apply this to the result from step 1. The function becomes: $y = f\left(-\frac{x}{2}\right)$.

The final function is $\boldsymbol{g(x) = f\left(-\frac{x}{2}\right)}$.

Ex 20: Consider a function $y = f(x)$. Find the equation of the resulting function, $g(x)$, if the graph of f is transformed by the following sequence:

1. A vertical stretch by a factor of 3.
2. A reflection in the x-axis.

Answer: We apply the transformations step-by-step to the initial function $y = f(x)$.

1. **Vertical stretch by a factor of 3**:
   This transformation multiplies the entire function by 3. The function becomes: $y = 3f(x)$.
2. **Reflection in the x-axis**:
   This transformation multiplies the current function by -1. We apply this to the result from step 1. The function becomes: $y = -(3f(x))$.

The final function is $\boldsymbol{g(x) = -3f(x)}$.

Ex 21: Consider a function $y = f(x)$. Find the equation of the resulting function, $g(x)$, if the graph of f is transformed by the following sequence:

1. A horizontal translation of 5 units to the left.
2. A horizontal compression by a factor of $\frac{1}{2}$.

Answer: We apply the transformations step-by-step to the initial function $y = f(x)$.

1. **Horizontal translation of 5 units to the left**:
   This transformation replaces x with $(x+5)$. The function becomes: $y = f(x+5)$.
2. **Horizontal compression by a factor of $\frac{1}{2}$**:
   This transformation replaces x with $2x$ in the current function. We apply this to the result from step 1. The function becomes: $y = f((2x)+5)$.

The final function is $\boldsymbol{g(x) = f(2x+5)}$.