EXPONENTS

Exponents are a short way to write repeated multiplication. They help us work with large numbers more easily.

A POSITIVE EXPONENTS

Definition Exponentiation

Exponentiation is repeated multiplication of a number by itself.
For a number a and a positive whole number n,

$$a^n = \overbrace{a \times a \times \cdots \times a}^{n \text{ factors}}.$$

$$\underset{\text{base}}{2}^{\overset{\text{exponent}}{4}} = \overbrace{2 \times 2 \times 2 \times 2}^{4 \text{ factors}}$$

Ex: Write using exponent notation: $5 \times 5 \times 5$.

Answer: $5 \times 5 \times 5 = 5^3$

Definition Vocabulary

Value	Expanded form	Exponent notation	Spoken form
2	2	2^1	2 or 2 to the power of 1
4	2×2	2^2	2 squared or 2 to the power of 2
8	$2 \times 2 \times 2$	2^3	2 cubed or 2 to the power of 3
16	$2 \times 2 \times 2 \times 2$	2^4	2 to the power of 4
32	$2 \times 2 \times 2 \times 2 \times 2$	2^5	2 to the power of 5

Ex: Find the value of 2^3.

Answer:

$$\begin{aligned} 2^3 &= 2 \times 2 \times 2 \\ &= 8 \end{aligned}$$

B NEGATIVE EXPONENTS

Definition Exponentiation for a negative exponent

For a non-zero number a and a *positive integer* n, we extend exponentiation to negative exponents by:

$$\begin{aligned} a^{-n} &= \frac{1}{\underbrace{a \times a \times \cdots \times a}_{n \text{ factors}}} \qquad \text{and} \qquad a^0 = 1 \quad (a \neq 0). \\ &= \frac{1}{a^n} \end{aligned}$$

In particular, $a^{-1} = \dfrac{1}{a}$. A negative exponent means we take the reciprocal of the corresponding positive power.

Ex: Write 3^{-2} as a fraction.

Answer:

$$\begin{aligned} 3^{-2} &= \frac{1}{3 \times 3} \\ &= \frac{1}{9} \end{aligned}$$

C RATIONAL EXPONENTS

Definition Rational Exponent

For a positive number a and positive integers m and n,

$$a^{\frac{1}{2}} = \sqrt{a},$$
$$a^{\frac{1}{n}} = \sqrt[n]{a},$$
$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.$$

Ex 1: Write $\sqrt{5}$ in exponent form.

Answer:

$$\sqrt{5} = 5^{\frac{1}{2}}$$

D EXPONENT LAW 1

Proposition Exponent Law 1

When we multiply two powers with the same base, we keep the base and add the exponents:

$$a^m \times a^n = a^{m+n}.$$

Ex: Simplify $5^2 \times 5^4$.

Answer:

$$\begin{aligned} 5^2 \times 5^4 &= 5^{2+4} \quad \text{(same base, add exponents)} \\ &= 5^6. \end{aligned}$$

E EXPONENT LAW 2

Proposition Exponent Law 2

For $a \neq 0$ and any numbers m and n,

$$\frac{a^m}{a^n} = a^{m-n}$$

Ex: Simplify $\frac{5^7}{5^3}$.

Answer:

$$\begin{aligned} \frac{5^7}{5^3} &= 5^{7-3} \\ &= 5^4 \end{aligned}$$

F EXPONENT LAW 3

Proposition Exponent Law 3

For $a \neq 0$ and any numbers m and n,

$$\left(a^m\right)^n = a^{m \times n}$$

Ex: Simplify $\left(5^2\right)^5$.

Answer:

$$\begin{aligned} \left(5^2\right)^5 &= 5^{2 \times 5} \\ &= 5^{10} \end{aligned}$$

G EXPONENT LAW 4

Proposition Exponent Law 4

For any numbers n and any numbers a and b,

$$(ab)^n = a^n b^n$$

Ex: Simplify $(2 \times 5)^3$.

Answer:

$$(2 \times 5)^3 = 2^3 5^3$$

H EXPONENT LAW 5

Proposition Exponent Law 5

For $b \neq 0$ and any number n,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Ex: Calculate $\left(\frac{5}{3}\right)^2$.

Answer:

$$\begin{aligned}\left(\frac{5}{3}\right)^2 &= \frac{5^2}{3^2}\\ &= \frac{25}{9}\end{aligned}$$

I EXPONENT LAW 6

Proposition Exponent Law 6

For non-zero numbers a and b, and any number n,

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

and in particular,

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}.$$

Ex: Calculate $\left(\frac{5}{3}\right)^{-2}$.

Answer:

$$\begin{aligned}\left(\frac{5}{3}\right)^{-2} &= \left(\frac{3}{5}\right)^2\\ &= \frac{3^2}{5^2}\\ &= \frac{9}{25}\end{aligned}$$

J ORDER OF OPERATIONS

The order of operations is a set of rules that tells us which calculations to do first in a mathematical expression.

Definition Order of Operations

To solve mathematical expressions accurately, we follow the **order of operations**, which is commonly remembered using the acronym **PEMDAS**:

1. P: Parentheses
2. E: Exponents
3. M: Multiplication
4. D: Division
5. A: Addition
6. S: Subtraction

We first do the operations at the top of the list.Multiplication and division are on the same level, so we work *from left to right*.Addition and subtraction are also on the same level, so we again work *from left to right*.

Ex: Evaluate $(1+2)\times 2^3+4$.

Answer:

$$\begin{aligned}(1+2)\times 2^3+4 &= (1+2)\times 2^3+4 && \text{(parentheses: } (1+2)=3) \\ &= 3\times 2^3+4 && \text{(exponent: } 2^3=8) \\ &= 3\times 8+4 && \text{(multiplication: } 3\times 8=24) \\ &= 24+4 && \text{(addition: } 24+4=28) \\ &= 28\end{aligned}$$

K SCIENTIFIC NOTATION

Working with very large or very small numbers can be awkward. Since our number system is base ten, we can use powers of ten to rewrite very large or very small numbers to make them easier to work with. This way of writing numbers is called **scientific notation** and is especially useful in science.

Definition Scientific Notation

A non-zero number is expressed in **scientific notation** when it is written in the form:

$$a\times 10^n \text{ where } 1\leq |a|<10 \text{ and } n \text{ is an integer.}$$

Ex: Write 245 in scientific notation.

Answer:

$$\begin{aligned}245 &= 2.45\times 100 \\ &= 2.45\times 10^2\end{aligned}$$

So 245 in scientific notation is 2.45×10^2.

L EXPONENTIAL EXPRESSION

Definition Exponential Expression

An **exponential expression** is a mathematical expression where a variable appears in the exponent.

Ex: 2^x and 5^{x+1} are exponential expressions. This is different from a polynomial expression like x^2, where the variable is in the base.

Method Manipulating Exponential Expressions

Applying the exponent laws allows us to simplify, expand, and factorize complex expressions involving variables in the exponent. These skills are fundamental for solving exponential equations.

Ex: Simplify $\frac{2^{x+1}+2^x}{2^x}$.

Answer:

$$\begin{aligned}\frac{2^{x+1}+2^x}{2^x} &= \frac{2^x\cdot 2^1+2^x}{2^x} && \text{(Using exponent law)} \\ &= \frac{2^x(2+1)}{2^x} && \text{(Factor out the common term } 2^x) \\ &= 3 && \text{(Cancel the common factor)}\end{aligned}$$

M THE EXPONENTIAL NUMBER e

Definition The Exponential Number e

The number **e** is defined as

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n .$$

Correct to five decimal places,

$$e \approx 2.71828\ldots$$

It is an irrational number, which means its decimal representation never terminates and never repeats.

N EXPONENTIAL EQUATIONS

Definition Exponential Equation

An **exponential equation** is an equation in which the unknown occurs as part of the index or exponent.

Ex: $2^x = 8$ and $30 \times 3^x = 7$ are both exponential equations.

There are a number of methods we can use to solve exponential equations. These include graphing, using technology, and by using logarithms, which we will study later. However, in some cases we can solve the equation algebraically by equating indices.

Method Solving by Equating Indices

For $a > 0, a \neq 1$, $a^x = a^y$ if and only if $x = y$.

Ex: Solve for x:

$$2^x = 16$$

Answer:

$$\begin{aligned} & 2^x = 16 \\ \Leftrightarrow\ & 2^x = 2^4 \quad \text{(Write 16 as a power of 2)} \\ \Leftrightarrow\ & x = 4 \quad \text{(Equate the indices)} \end{aligned}$$