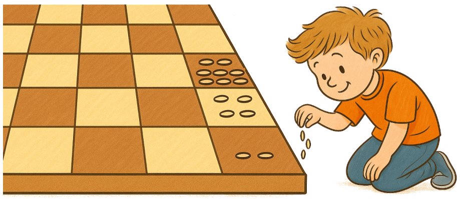
EXPONENTS

Exponents are a short way to write repeated multiplication. They help us work with large numbers more easily.

A POSITIVE EXPONENTS

Discover: Imagine you have a chessboard. You place two grains of wheat on the first square, four grains on the second square, eight grains on the third square, and so on, doubling the number of grains on each next square.



How many grains of wheat are on the last square of a chessboard with 64 squares?

Answer:

Square number	Number of grains
1	2
2	2×2
3	$2 \times 2 \times 2$
$\vdots$	$\vdots$
64	$\overbrace{2 \times 2 \times \cdots \times 2}^{64 \text{ factors}}$

Rather than writing $\overbrace{2 \times 2 \times \cdots \times 2}^{64 \text{ factors}}$, we can write this product as 2^{64}.
This means there are 2^{64} grains on the last square. Using a calculator:

$$2^{64} = 18\,446\,744\,073\,709\,551\,616.$$

This is an enormous number!

Definition Exponentiation

Exponentiation is repeated multiplication of a number by itself.
For a number a and a positive whole number n,

$$a^n = \overbrace{a \times a \times \cdots \times a}^{n \text{ factors}}.$$

exponent

$$\text{base} \rightarrow 2^4 = \overbrace{2 \times 2 \times 2 \times 2}^{4 \text{ factors}}$$

Ex: Write using exponent notation: $5 \times 5 \times 5$.

Answer: $5 \times 5 \times 5 = 5^3$

Definition Vocabulary

Value	Expanded form	Exponent notation	Spoken form
2	2	2^1	2 or 2 to the power of 1
4	2×2	2^2	2 squared or 2 to the power of 2
8	$2 \times 2 \times 2$	2^3	2 cubed or 2 to the power of 3
16	$2 \times 2 \times 2 \times 2$	2^4	2 to the power of 4
32	$2 \times 2 \times 2 \times 2 \times 2$	2^5	2 to the power of 5

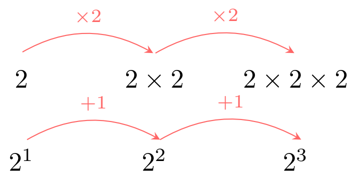
Ex: Find the value of 2^3.

Answer:

$$\begin{aligned} 2^3 &= 2 \times 2 \times 2 \\ &= 8 \end{aligned}$$

B NEGATIVE EXPONENTS

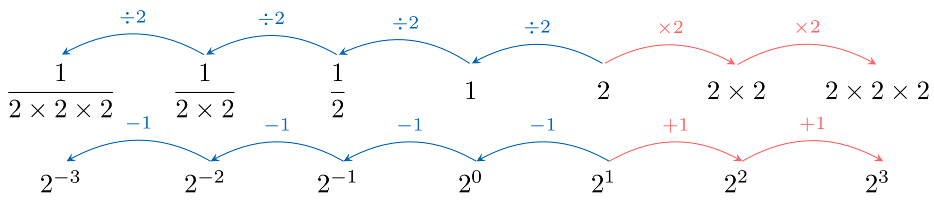
Discover: To understand negative exponents, let's explore the pattern of multiplying by 2:



From this pattern, you can see:

- $2^1 = 2$
- $2^2 = 2 \times 2$
- $2^3 = 2 \times 2 \times 2$

Because division is the inverse of multiplication, we can divide by 2 repeatedly to extend the pattern:



From this extended pattern, you can see:

- $2^0 = 1$
- $2^{-1} = \frac{1}{2}$
- $2^{-2} = \frac{1}{2 \times 2}$
- $2^{-3} = \frac{1}{2 \times 2 \times 2}$

Definition Exponentiation for a negative exponent

For a non-zero number a and a *positive integer* n, we extend exponentiation to negative exponents by:

$$a^{-n} = \frac{1}{\underbrace{a \times a \times \cdots \times a}_{n \text{ factors}}} = \frac{1}{a^n} \qquad \text{and} \qquad a^0 = 1 \quad (a \neq 0).$$

In particular, $a^{-1} = \frac{1}{a}$. A negative exponent means we take the reciprocal of the corresponding positive power.

Ex: Write 3^{-2} as a fraction.

Answer:

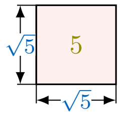
$$3^{-2} = \frac{1}{3 \times 3} = \frac{1}{9}$$

C RATIONAL EXPONENTS

Discover: We know about positive exponents, like $5^3 = 5 \times 5 \times 5$, and also about negative exponents, like $5^{-3} = \dfrac{1}{5 \times 5 \times 5}$.
But what about **fractional exponents**?
Using the exponent laws, let's see what happens with $5^{\frac{1}{2}}$:

$$\begin{aligned} 5^{\frac{1}{2}} \times 5^{\frac{1}{2}} &= 5^{\frac{1}{2}+\frac{1}{2}} \\ &= 5^1 \\ &= 5 \end{aligned}$$

And by the definition of the square root:



$$\sqrt{5} \times \sqrt{5} = 5$$

By comparing these two results, we see that:

$$5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = \sqrt{5} \times \sqrt{5}$$

So, we can deduce that:

$$5^{\frac{1}{2}} = \sqrt{5}$$

In this chapter, we will only use rational exponents with **positive bases**, so that roots like $\sqrt[n]{a}$ are real numbers. This shows us that we can use **fractional exponents** to represent roots, extending our understanding of exponents to include *rational exponents*.

Definition Rational Exponent

For a positive number a and positive integers m and n,

$$\begin{aligned} a^{\frac{1}{2}} &= \sqrt{a}, \\ a^{\frac{1}{n}} &= \sqrt[n]{a}, \\ a^{\frac{m}{n}} &= \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m. \end{aligned}$$

Ex 1: Write $\sqrt{5}$ in exponent form.

Answer:

$$\sqrt{5} = 5^{\frac{1}{2}}$$

D EXPONENT LAW 1

Discover: Let's look at an example:

$$\begin{aligned} 7^3 \times 7^2 &= \overbrace{7 \times 7 \times 7}^{3 \text{ factors}} \times \overbrace{7 \times 7}^{2 \text{ factors}} \\ &= \overbrace{7 \times 7 \times 7 \times 7 \times 7}^{3+2 \text{ factors}} \\ &= 7^{3+2}. \end{aligned}$$

In this example we are multiplying two powers with the same base (7).
We can see that we keep the base and add the exponents: $3 + 2 = 5$.
In general, when a number a is raised to the power m and multiplied by the same number raised to the power n, that is

$$a^m \times a^n,$$

the result is equal to a raised to the sum of the exponents:

$$a^m \times a^n = a^{m+n}.$$

Proposition Exponent Law 1

When we multiply two powers with the same base, we keep the base and add the exponents:

$$a^m \times a^n = a^{m+n}.$$

Proof

$$a^m \times a^n = \overbrace{a \times \cdots \times a}^{m \text{ factors}} \times \overbrace{a \times \cdots \times a}^{n \text{ factors}}$$
$$= \overbrace{a \times \cdots \times a}^{m+n \text{ factors}}$$
$$= a^{m+n}$$

Ex: Simplify $5^2 \times 5^4$.

Answer:

$$5^2 \times 5^4 = 5^{2+4} \quad \text{(same base, add exponents)}$$
$$= 5^6.$$

E EXPONENT LAW 2

Discover: Let's look at an example:

$$\frac{7^5}{7^2} = \frac{\overbrace{\cancel{7} \times \cancel{7} \times 7 \times 7 \times 7}^{5 \text{ factors}}}{\underbrace{\cancel{7} \times \cancel{7}}_{2 \text{ factors}}}$$
$$= \overbrace{7 \times 7 \times 7}^{5-2 \text{ factors}}$$
$$= 7^{5-2}$$

In general, when a number a is raised to the power m and divided by the same number raised to the power n, that is

$$\frac{a^m}{a^n},$$

the result is a raised to the difference of the exponents:

$$\frac{a^m}{a^n} = a^{m-n}.$$

Proposition **Exponent Law 2**

For $a \neq 0$ and any numbers m and n,

$$\frac{a^m}{a^n} = a^{m-n}$$

Ex: Simplify $\frac{5^7}{5^3}$.

Answer:

$$\frac{5^7}{5^3} = 5^{7-3}$$
$$= 5^4$$

F EXPONENT LAW 3

Discover: Let's look at an example:

$$(5^2)^3 = (\overbrace{5 \times 5}^{2 \text{ factors}})^3$$
$$= \overbrace{(\overbrace{5 \times 5}^{2 \text{ factors}}) \times (\overbrace{5 \times 5}^{2 \text{ factors}}) \times (\overbrace{5 \times 5}^{2 \text{ factors}})}^{3 \text{ factors}}$$
$$= 5^{2+2+2}$$
$$= 5^{2\times 3}$$

In general, when a number a is raised to the power m, and that result is raised to the power n, that is

$$(a^m)^n,$$

the result is a raised to the product of the exponents:

$$(a^m)^n = a^{m\times n}.$$

Proposition Exponent Law 3

For $a \neq 0$ and any numbers m and n,

$$(a^m)^n = a^{m \times n}$$

Ex: Simplify $\left(5^2\right)^5$.

Answer:

$$\begin{aligned}\left(5^2\right)^5 &= 5^{2\times 5}\\ &= 5^{10}\end{aligned}$$

G EXPONENT LAW 4

Discover: Let's look at an example:

$$\begin{aligned}(3 \times 5)^2 &= (3 \times 5) \times (3 \times 5)\\ &= 3 \times 5 \times 3 \times 5\\ &= (3 \times 3) \times (5 \times 5)\\ &= 3^2\, 5^2\end{aligned}$$

In general, when you multiply two numbers a and b, and then raise the product to the power n, that is

$$(ab)^n,$$

the result is each factor raised to the power n:

$$(ab)^n = a^n\, b^n.$$

Proposition Exponent Law 4

For any numbers n and any numbers a and b,

$$(ab)^n = a^n\, b^n$$

Ex: Simplify $(2 \times 5)^3$.

Answer:

$$(2 \times 5)^3 = 2^3\, 5^3$$

H EXPONENT LAW 5

Discover: Let's look at an example:

$$\begin{aligned}\left(\frac{5}{3}\right)^2 &= \left(\frac{5}{3}\right) \times \left(\frac{5}{3}\right)\\ &= \frac{5 \times 5}{3 \times 3}\\ &= \frac{5^2}{3^2}\end{aligned}$$

In general, when a quotient $\frac{a}{b}$ is raised to a power n, that is

$$\left(\frac{a}{b}\right)^n,$$

the result is the numerator raised to that power divided by the denominator raised to that power:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Proposition Exponent Law 5

For $b \neq 0$ and any number n,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Ex: Calculate $\left(\frac{5}{3}\right)^2$.

Answer:

$$\begin{aligned}\left(\frac{5}{3}\right)^2 &= \frac{5^2}{3^2}\\ &= \frac{25}{9}\end{aligned}$$

I EXPONENT LAW 6

Discover: Let's look at an example with a negative exponent:

$$\begin{aligned}\left(\frac{5}{3}\right)^{-2} &= \frac{1}{\left(\frac{5}{3}\right)^{2}}\\ &= \frac{1}{\frac{5^2}{3^2}}\\ &= 1\times\frac{3^2}{5^2}\\ &= \frac{3^2}{5^2}\\ &= \left(\frac{3}{5}\right)^{2}\end{aligned}$$

In general, when a quotient $\frac{a}{b}$ is raised to a negative power $-n$,

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}.$$

This means that a negative exponent makes the fraction *flip*: the numerator and denominator swap places.

Proposition Exponent Law 6

For non-zero numbers a and b, and any number n,

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$$

and in particular,

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}.$$

Ex: Calculate $\left(\frac{5}{3}\right)^{-2}$.

Answer:

$$\begin{aligned}\left(\frac{5}{3}\right)^{-2} &= \left(\frac{3}{5}\right)^{2}\\ &= \frac{3^2}{5^2}\\ &= \frac{9}{25}\end{aligned}$$

J ORDER OF OPERATIONS

The order of operations is a set of rules that tells us which calculations to do first in a mathematical expression.

Definition Order of Operations

To solve mathematical expressions accurately, we follow the **order of operations**, which is commonly remembered using the acronym **PEMDAS**:

1. P: Parentheses
2. E: Exponents
3. M: Multiplication
4. D: Division
5. A: Addition
6. S: Subtraction

We first do the operations at the top of the list.Multiplication and division are on the same level, so we work *from left to right*.Addition and subtraction are also on the same level, so we again work *from left to right*.

Ex: Evaluate $(1+2)\times 2^3+4$.

Answer:

$$\begin{aligned}(1+2)\times 2^3+4 &= (1+2)\times 2^3+4 && \text{(parentheses: } (1+2)=3)\\ &= 3\times 2^3+4 && \text{(exponent: } 2^3=8)\\ &= 3\times 8+4 && \text{(multiplication: } 3\times 8=24)\\ &= 24+4 && \text{(addition: } 24+4=28)\\ &= 28\end{aligned}$$

K SCIENTIFIC NOTATION

Working with very large or very small numbers can be awkward. Since our number system is base ten, we can use powers of ten to rewrite very large or very small numbers to make them easier to work with. This way of writing numbers is called **scientific notation** and is especially useful in science.

Definition Scientific Notation

A non-zero number is expressed in **scientific notation** when it is written in the form:

$$a\times 10^n \text{ where } 1\le |a|<10 \text{ and } n \text{ is an integer.}$$

Ex: Write 245 in scientific notation.

Answer:

$$\begin{aligned}245 &= 2.45\times 100\\ &= 2.45\times 10^2\end{aligned}$$

So 245 in scientific notation is 2.45×10^2.

L EXPONENTIAL EXPRESSION

Definition Exponential Expression

An **exponential expression** is a mathematical expression where a variable appears in the exponent.

Ex: 2^x and 5^{x+1} are exponential expressions. This is different from a polynomial expression like x^2, where the variable is in the base.

Method Manipulating Exponential Expressions

Applying the exponent laws allows us to simplify, expand, and factorize complex expressions involving variables in the exponent. These skills are fundamental for solving exponential equations.

Ex: Simplify $\frac{2^{x+1}+2^x}{2^x}$.

Answer:

$$\begin{aligned}\frac{2^{x+1}+2^x}{2^x} &= \frac{2^x\cdot 2^1+2^x}{2^x} && \text{(Using exponent law)}\\ &= \frac{2^x(2+1)}{2^x} && \text{(Factor out the common term } 2^x)\\ &= 3 && \text{(Cancel the common factor)}\end{aligned}$$

M THE EXPONENTIAL NUMBER e

Discover: Let's explore an idea from finance: compound interest.
Imagine you invest \$1 in a special bank account that offers a 100% annual interest rate. We will see how much money you have after one year, depending on how often the interest is calculated (compounded) and added to your account.

- **Case 1: Compounded annually**
  The interest is paid once at the end of the year. The value is:
  $$1\times(1+100\%) = 1\times(1+1) = \$2.$$
  So
  $$\text{Value} = (1+1)^1.$$
- **Case 2: Compounded semi-annually**
  The interest is paid twice a year. The bank gives you half the annual rate (50%) each time.

- After 6 months: $1 \times \left(1 + \frac{1}{2}\right) = \$1.50$
- At the end of the year: $1.50 \times \left(1 + \frac{1}{2}\right) = \$2.25$

This can be calculated in one step as

$$1 \times \left(1 + \frac{1}{2}\right)^2 = \$2.25.$$

So

$$\text{Value} = \left(1 + \frac{1}{2}\right)^2.$$

- **Case n: Compounded n times per year**

  If we compound n times per year, the interest rate per period is $\frac{1}{n}$ (since the total annual rate is 100%), and it is applied n times. The value after one year is

$$\text{Value} = \left(1 + \frac{1}{n}\right)^n.$$

Let's calculate the value of $\left(1 + \frac{1}{n}\right)^n$ for increasingly large values of n:

Compounding Frequency	n	**Value** $\left(1 + \frac{1}{n}\right)^n$
Annually	1	$(1 + 1/1)^1 = 2.00000$
Semi-annually	2	$(1 + 1/2)^2 = 2.25000$
Quarterly	4	$(1 + 1/4)^4 \approx 2.44141$
Monthly	12	$(1 + 1/12)^{12} \approx 2.61304$
Daily	365	$(1 + 1/365)^{365} \approx 2.71457$
Hourly	8,760	$(1 + 1/8760)^{8760} \approx 2.71813$

As you can see, the final amount increases, but it does not grow without bound. It seems to approach a specific number. This special irrational number is denoted by e, also known as Euler's number. It represents the limiting value of this process of increasingly frequent compounding.

Definition The Exponential Number e

The number **e** is defined as

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Correct to five decimal places,

$$e \approx 2.71828\ldots$$

It is an irrational number, which means its decimal representation never terminates and never repeats.

N EXPONENTIAL EQUATIONS

Definition Exponential Equation

An **exponential equation** is an equation in which the unknown occurs as part of the index or exponent.

Ex: $2^x = 8$ and $30 \times 3^x = 7$ are both exponential equations.

There are a number of methods we can use to solve exponential equations. These include graphing, using technology, and by using logarithms, which we will study later. However, in some cases we can solve the equation algebraically by equating indices.

Method Solving by Equating Indices

For $a > 0, a \neq 1$, $a^x = a^y$ if and only if $x = y$.

Ex: Solve for x:

$$2^x = 16$$

Answer:

$$\begin{aligned} & 2^x = 16 \\ \Leftrightarrow\ & 2^x = 2^4 && \text{(Write 16 as a power of 2)} \\ \Leftrightarrow\ & x = 4 && \text{(Equate the indices)} \end{aligned}$$