EXPONENTIAL FUNCTIONS

In previous chapters, we learned how to evaluate expressions like a^n, where the exponent n was an integer or a rational number. This chapter extends that concept to the **exponential function**, written as $f(x) = a^x$, where the exponent x can be any real number.
We will explore the key features and graphs of these functions and see how they are used to model real-world phenomena involving rapid growth or decay, such as population dynamics and compound interest.

A EXPONENTIAL FUNCTIONS

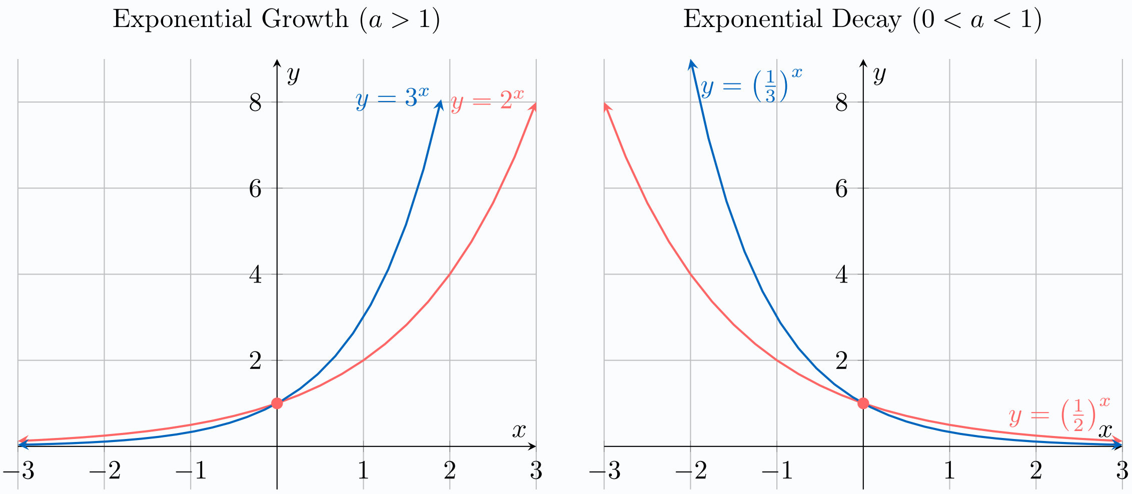
Definition Exponential Function

The **exponential function** has the form $f(x) = a^x$, where the base a is a positive constant and $a \neq 1$.

Proposition Key Features of the Graph of $y = a^x$

All exponential functions of the form $f(x) = a^x$ share several key graphical features:

- **Domain:** The domain is all real numbers, $(-\infty, \infty)$.
- **Range:** The range is all positive real numbers, $(0, \infty)$.
- **Horizontal Asymptote:** The graph has a horizontal asymptote at the x-axis ($y = 0$). The function approaches this line but never touches it.
- **y-intercept:** The graph always passes through the point $(0, 1)$, because $a^0 = 1$ for any valid base a.
- **General Shape:** The shape of the graph is determined by the value of the base, a:
  - If $a > 1$, the function shows **exponential growth** and is increasing.
  - If $0 < a < 1$, the function shows **exponential decay** and is decreasing.



B NATURAL EXPONENTIAL FUNCTION e^x

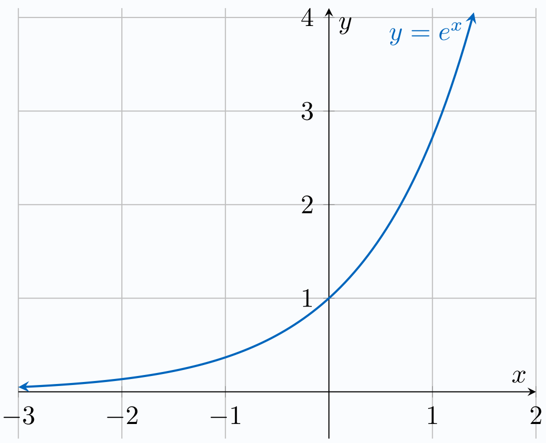
Definition Natural Exponential Function

The **natural exponential function** is $x \mapsto e^x$.

- Domain: $(-\infty, +\infty)$
- Range: $(0, +\infty)$

Proposition Graph and Properties of $y = e^x$

- **Horizontal Asymptote:** The graph has a horizontal asymptote at the x-axis ($y = 0$) as $x \to -\infty$.
- **y-intercept:** The graph passes through the point $(0, 1)$, since $e^0 = 1$.
- It is a strictly **increasing** function.



C TRANSFORMATIONS OF EXPONENTIAL FUNCTIONS

Proposition Transformations of Exponential Functions

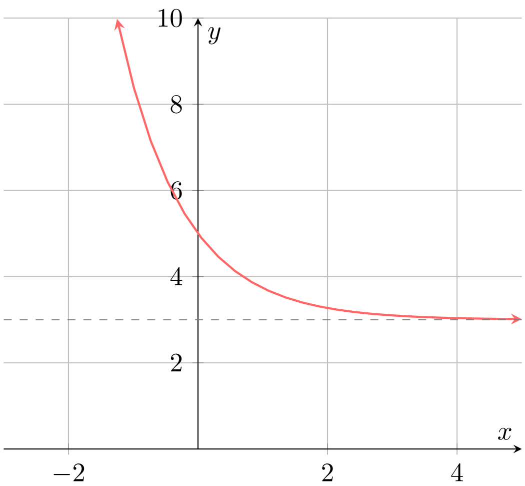
The graph of the general exponential function $f(x) = k \cdot a^{m(x-h)} + v$ can be obtained by applying transformations to the basic graph of $y = a^x$.

- **Vertical Translation (v):** The graph is shifted up by v units. The horizontal asymptote becomes $y = v$.
- **Horizontal Translation (h):** The graph is shifted to the right by h units.
- **Vertical Stretch/Reflection (k):** The graph is stretched vertically by a factor of $|k|$. If $k < 0$, the graph is reflected in the horizontal asymptote.
- **Horizontal Stretch/Reflection (m):** The graph is stretched horizontally by a factor of $1/|m|$ about the line $x = h$. If $m < 0$, it is reflected across the vertical line $x = h$ (across the y-axis only when $h = 0$).

Ex: Sketch the graph of $f(x) = 2e^{-x} + 3$. State the domain, range, and equation of the asymptote.

Answer: The graph is a transformation of $y = e^x$:

- Reflection in the y-axis (due to the negative in front of x).
- Vertical stretch by a factor of 2.
- Vertical shift 3 units up.
- The horizontal asymptote is $\boldsymbol{y = 3}$.
- The domain is $\mathbb{R}$.
- The range is $\mathbf{(3, \infty)}$.



D EXPONENTIAL MODELS

Exponential functions are used to model quantities that change by a **constant multiplicative factor** over equal intervals of time. This core principle distinguishes them from linear functions, which change by a constant difference (addition or subtraction).

There are two main types of exponential models:

- **Exponential Growth:** The quantity increases by a constant factor greater than 1. This is seen in phenomena like population growth and compound interest.
- **Exponential Decay:** The quantity decreases by a constant factor between 0 and 1. This is seen in phenomena like radioactive decay and asset depreciation.

These models can be **discrete**, occurring in distinct steps (e.g., interest compounded annually), or **continuous**, occurring smoothly over time (e.g., bacterial growth).

Definition General Model for Exponential Growth and Decay

An exponential relationship is described by the function:

$$A(t) = A_0 \times R^t$$

where:

- $A(t)$ is the amount at time t.
- A_0 is the initial amount (the amount at $t = 0$).
- R is the constant growth or decay factor per unit of time.
- t is the time elapsed.

Ex: The population of foxes, P, in a specified area, t years after observation began, is modeled by the equation: $P(t) = 300(1.25)^t$.

1. How many foxes are there initially?
2. What is the annual percentage growth rate?
3. How many foxes are there after 5 years?

Answer:

1. The initial population corresponds to $t = 0$.
$$P(0) = 300(1.25)^0 = 300 \times 1 = \mathbf{300} \text{ foxes}$$
2. The growth factor is $R = 1.25$. Since $R = 1 + r$, we have $1.25 = 1 + r$, which gives $r = 0.25$.
The annual growth rate is **25%**.
3. Substitute $t = 5$ into the equation. Since the population must be a whole number, we round to the nearest fox.
$$P(5) = 300(1.25)^5 \approx 915.52... \approx \mathbf{916} \text{ foxes}$$

Ex: An amount of \$5 000 is invested at 6% p.a. compounded annually.

1. Find a model for the amount, A, after t years.
2. Find the amount after 4 years.

Answer:

1. The initial amount is $A_0 = 5\,000$. The annual interest rate is $r = 0.06$.
The growth factor is $R = 1 + r = 1 + 0.06 = 1.06$.
The model is $\boldsymbol{A(t) = 5\,000(1.06)^t}$.
2. After 4 years, the amount is:
$$A(4) = 5\,000(1.06)^4 \approx 6\,312.38...$$
The amount is **\$6 312.38**.