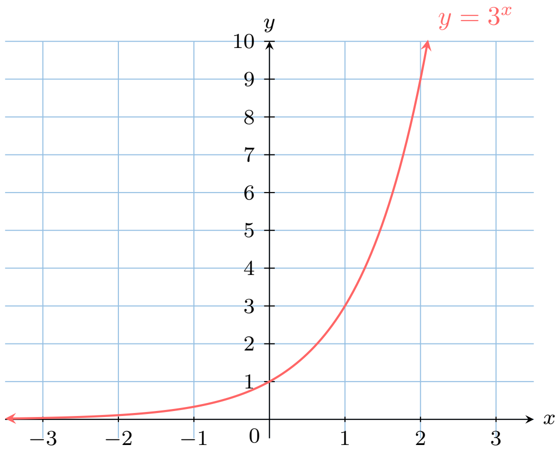
EXPONENTIAL FUNCTIONS

A EXPONENTIAL FUNCTIONS

A.1 READING AND SKETCHING EXPONENTIAL FUNCTIONS

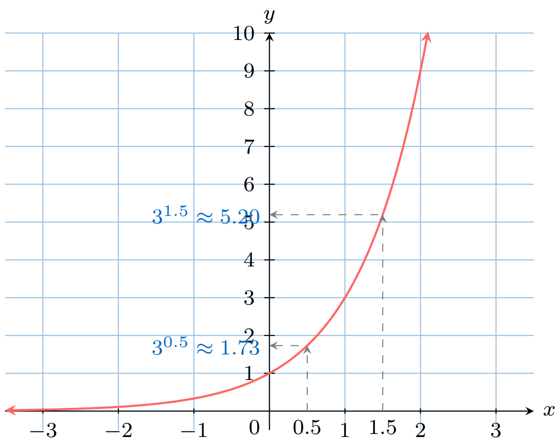
Ex 1:



By reading from the graph of $y = 3^x$, complete the following inequalities with consecutive integers:

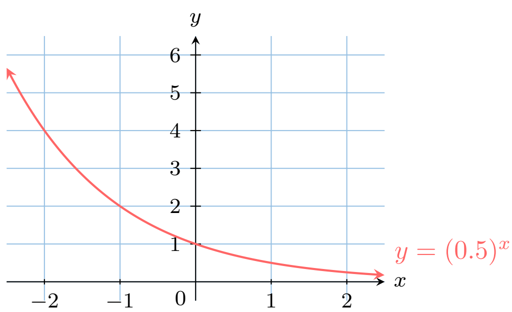
1. $\boxed{5} \leq 3^{1.5} < \boxed{6}$
2. $\boxed{1} \leq 3^{0.5} < \boxed{2}$

Answer: The values can be estimated visually using the graph above:



1. We read $3^{1.5} \approx 5.2$, so the answer is **between 5 and 6.**
$$\boxed{5} \leq 3^{1.5} < \boxed{6}$$
2. We read $3^{0.5} \approx 1.7$, so the answer is **between 1 and 2.**
$$\boxed{1} \leq 3^{0.5} < \boxed{2}$$

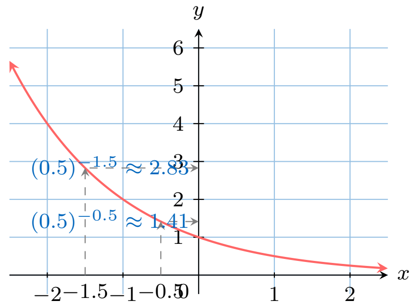
Ex 2:



By reading from the graph of $y = (0.5)^x$, complete the following inequalities with consecutive integers:

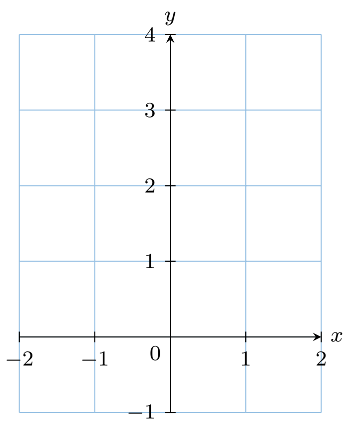
1. $\boxed{1} \leq (0.5)^{-0.5} < \boxed{2}$
2. $\boxed{2} \leq (0.5)^{-1.5} < \boxed{3}$

Answer: The values can be estimated visually using the graph above:



1. We read $(0.5)^{-0.5} \approx 1.41$, so the answer is **between 1 and 2.**
$$\boxed{1} \leq (0.5)^{-0.5} < \boxed{2}$$
2. We read $(0.5)^{-1.5} \approx 2.83$, so the answer is **between 2 and 3.**
$$\boxed{2} \leq (0.5)^{-1.5} < \boxed{3}$$

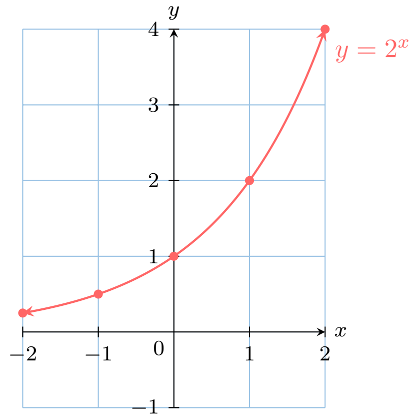
Ex 3: Sketch the graph of the function $f(x) = 2^x$. Use a table of values for integer values of x from -2 to 2 to help you plot key points.



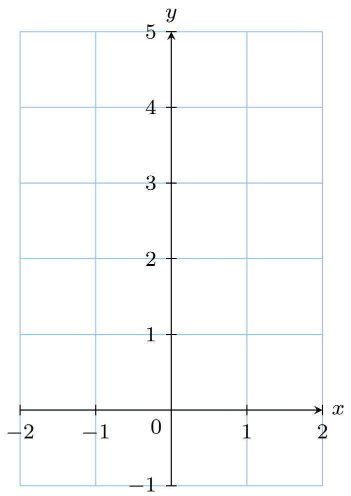
Answer: Fill in the table of values:

x	-2	-1	0	1	2
$f(x)$	0.25	0.5	1	2	4

Plot the points and draw the graph:



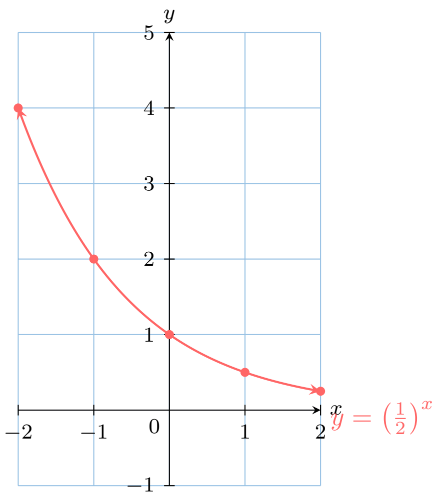
Ex 4: Sketch the graph of the function $f(x) = \left(\frac{1}{2}\right)^x$. Use a table of values for integer values of x from -2 to 2 to help you plot key points.



Answer: Fill in the table of values:

x	-2	-1	0	1	2
$f(x)$	4	2	1	0.5	0.25

Plot the points and draw the graph:



A.2 EVALUATING EXPONENTIAL FUNCTIONS

Ex 5: For $f(x) = 3^x$, evaluate:

1. $f(2) = \boxed{9}$
2. $f(0) = \boxed{1}$
3. $f(-1) = \boxed{\dfrac{1}{3}}$

Answer:

1. $f(2) = 3^2$
   $= 9$
2. $f(0) = 3^0$
   $= 1$
3. $f(-1) = 3^{-1}$
   $= \dfrac{1}{3^1}$
   $= \dfrac{1}{3}$

Ex 6: For $f(x) = 10^x$, evaluate:

1. $f(2) = \boxed{100}$
2. $f(0) = \boxed{1}$
3. $f(-1) = \boxed{\dfrac{1}{10}}$

Answer:

1. $f(2) = 10^2$
   $= 100$
2. $f(0) = 10^0$
   $= 1$
3. $f(-1) = 10^{-1}$
   $= \dfrac{1}{10^1}$
   $= \dfrac{1}{10}$

Ex 7: For $f(x) = \left(\frac{1}{2}\right)^x$, evaluate:

1. $f(-2) = \boxed{4}$
2. $f(-1) = \boxed{2}$
3. $f(0) = \boxed{1}$
4. $f(1) = \boxed{\dfrac{1}{2}}$

Answer:

1. $f(-2) = \left(\dfrac{1}{2}\right)^{-2}$
   $= \left(\dfrac{2}{1}\right)^{2}$
   $= 2^2$
   $= 4$

2. $f(-1) = \left(\frac{1}{2}\right)^{-1}$
$= \left(\frac{2}{1}\right)^{1}$
$= 2$

3. $f(0) = \left(\frac{1}{2}\right)^{0}$
$= 1$

4. $f(1) = \left(\frac{1}{2}\right)^{1}$
$= \frac{1}{2}$

B NATURAL EXPONENTIAL FUNCTION e^x

B.1 CALCULATING WITH THE NATURAL EXPONENTIAL FUNCTION

Ex 8: Using your calculator, evaluate the following values of $f(x) = e^x$. Round your answers to 2 decimal places.

1. $e^1 \approx \boxed{2.72}$
2. $e^2 \approx \boxed{7.39}$
3. $e^{-1} \approx \boxed{0.37}$
4. $\sqrt{e} \approx \boxed{1.65}$

Answer: Using the e^x button on the calculator:

1. $e^1 \approx 2.718... \approx \mathbf{2.72}$
2. $e^2 \approx 7.389... \approx \mathbf{7.39}$
3. $e^{-1} \approx 0.367... \approx \mathbf{0.37}$
4. $\sqrt{e} = e^{0.5} \approx 1.648... \approx \mathbf{1.65}$

Ex 9: Simplify the following expressions using the laws of exponents.

1. $e^x \cdot e^2 = \boxed{e^{x+2}}$
2. $\frac{e^{3x}}{e^x} = \boxed{e^{2x}}$
3. $(e^2)^x = \boxed{e^{2x}}$

Answer:

1. When multiplying terms with the same base, add the exponents: $e^x \cdot e^2 = e^{x+2}$.
2. When dividing terms with the same base, subtract the exponents: $\frac{e^{3x}}{e^x} = e^{3x-x} = e^{2x}$.
3. When raising a power to a power, multiply the exponents: $(e^2)^x = e^{2x}$.

Ex 10: Solve the following equations for x without using a calculator.

1. $e^x = 1 \implies x = \boxed{0}$
2. $e^x = e^5 \implies x = \boxed{5}$
3. $e^x = -3 \implies \boxed{\textbf{No solution}}$

Answer:

1. Since $e^0 = 1$, we have $x = 0$.
2. Since the bases are the same, the exponents must be equal: $x = 5$.
3. The range of e^x is $(0, \infty)$, meaning e^x is always positive. Therefore, $e^x = -3$ has **no solution**.

B.2 GRAPHING AND PROPERTIES OF THE NATURAL EXPONENTIAL FUNCTION

Ex 11: Based on the properties of the natural exponential function $y = e^x$:

1. What are the coordinates of the y-intercept?

$$(\boxed{0};\boxed{1})$$

2. What is the equation of the horizontal asymptote?

$$y = \boxed{0}$$

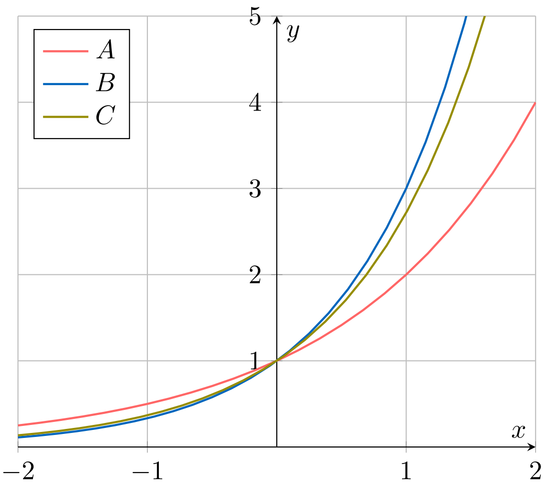
3. Is the function strictly increasing or decreasing?

$$\boxed{\textbf{Increasing}}$$

Answer:

1. Since $e^0 = 1$, the graph passes through $(\mathbf{0}, \mathbf{1})$.
2. As $x \to -\infty$, e^x gets closer to 0. The asymptote is $\mathbf{y = 0}$.
3. Since the base $e \approx 2.718 > 1$, the function is strictly **increasing**.

Ex 12: The graphs of $y = 2^x$, $y = 3^x$, and $y = e^x$ are plotted below. Identify which curve corresponds to $y = e^x$.



The curve representing $y = e^x$ is: $\boxed{\mathbf{C}}$

Answer: We know that $e \approx 2.718$.Comparing the bases:

$$2 < 2.718 < 3 \implies 2 < e < 3$$

Therefore, for $x > 0$, the graph of e^x must lie between the graphs of 2^x and 3^x.

- Curve A is 2^x (slowest growth).
- Curve B is 3^x (fastest growth).
- Curve C is $\boldsymbol{e^x}$ (middle growth).

Ex 13: Consider the function $g(x) = e^x - 2$.

1. Calculate the y-intercept of the graph of g.

$$y = \boxed{-1}$$

2. Determine the equation of the horizontal asymptote of g.

$$y = \boxed{-2}$$

Answer:

1. The y-intercept occurs when $x = 0$:

$$g(0) = e^0 - 2 = 1 - 2 = -1$$

2. The horizontal asymptote of e^x is $y = 0$. Since $g(x)$ is translated down by 2 units, the new asymptote is $y = 0 - 2$, so $\mathbf{y = -2}$.

C TRANSFORMATIONS OF EXPONENTIAL FUNCTIONS

C.1 CALCULATING $f(x)$

Ex 14: For $f : x \mapsto 3 \cdot 2^x$, find in simplest form:

1. $f(0) = \boxed{3}$
2. $f(2) = \boxed{12}$
3. $f(-1) = \boxed{\frac{3}{2}}$

Answer:

1. $f(0) = 3 \cdot 2^0 = 3 \cdot 1 = 3$
2. $f(2) = 3 \cdot 2^2 = 3 \cdot 4 = 12$
3. $f(-1) = 3 \cdot 2^{-1} = 3 \cdot \frac{1}{2} = \frac{3}{2}$

Ex 15: For $f : x \mapsto 5 \cdot e^x$, find in simplest form:

1. $f(0) = \boxed{5}$
2. $f(1) = \boxed{5e}$
3. $f(\ln 2) = \boxed{10}$

Answer:

1. $f(0) = 5 \cdot e^0 = 5 \cdot 1 = 5$
2. $f(1) = 5 \cdot e^1 = 5e$
3. $f(\ln 2) = 5 \cdot e^{\ln 2} = 5 \cdot 2 = 10$

Ex 16: For $f : x \mapsto \left(\frac{1}{3}\right)^x + 1$, find in simplest form:

1. $f(0) = \boxed{2}$
2. $f(2) = \boxed{10/9}$
3. $f(-2) = \boxed{10}$

Answer:

1. $f(0) = \left(\frac{1}{3}\right)^0 + 1 = 1 + 1 = 2$
2. $f(2) = \left(\frac{1}{3}\right)^2 + 1 = \frac{1}{9} + 1 = \frac{10}{9}$
3. $f(-2) = \left(\frac{1}{3}\right)^{-2} + 1 = 3^2 + 1 = 9 + 1 = 10$

C.2 FINDING $f(g(x))$

Ex 17: For the function $f(x) = x + 1$ and $g(x) = 4^x$, find and simplify:

$$(f \circ g)(x) = \boxed{4^x + 1}$$

Answer:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(4^x) \\ &= 4^x + 1\end{aligned}$$

Ex 18: For the function $f(x) = x + 1$ and $g(x) = 4^x$, find and simplify:

$$(g \circ f)(x) = \boxed{4^{x+1}}$$

Answer:

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x + 1) \\ &= 4^{x+1}\end{aligned}$$

Ex 19: For the function $f(x) = 3x$ and $g(x) = e^x$, find and simplify:

$$(f \circ g)(x) = \boxed{3e^x}$$

Answer:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(e^x) \\ &= 3e^x\end{aligned}$$

Ex 20: For the function $f(x) = 3x$ and $g(x) = e^x$, find and simplify:

$$(g \circ f)(x) = \boxed{e^{3x}}$$

Answer:

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x) \\ &= e^{3x}\end{aligned}$$

Ex 21: For the function $f(x) = x - 2$ and $g(x) = 5^x$, find and simplify:

$$(f \circ g)(x) = \boxed{5^x - 2}$$

Answer:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(5^x) \\ &= 5^x - 2\end{aligned}$$

Ex 22: For the function $f(x) = x - 2$ and $g(x) = 5^x$, find and simplify:

$$(g \circ f)(x) = \boxed{5^{x-2}}$$

Answer:

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x - 2) \\ &= 5^{x-2}\end{aligned}$$

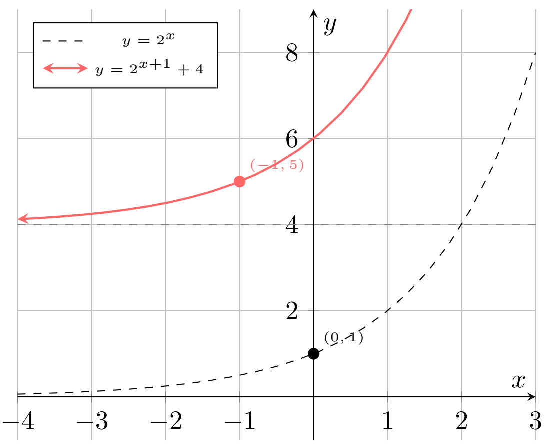
C.3 DESCRIBING TRANSFORMATIONS OF EXPONENTIAL GRAPHS

Ex 23: Describe the sequence of transformations that maps the graph of $y = 2^x$ to the graph of $f(x) = 2^{x+1} + 4$.

Answer: The function $f(x) = 2^{x+1} + 4$ is a transformation of the base function $y = 2^x$.
The transformations are:

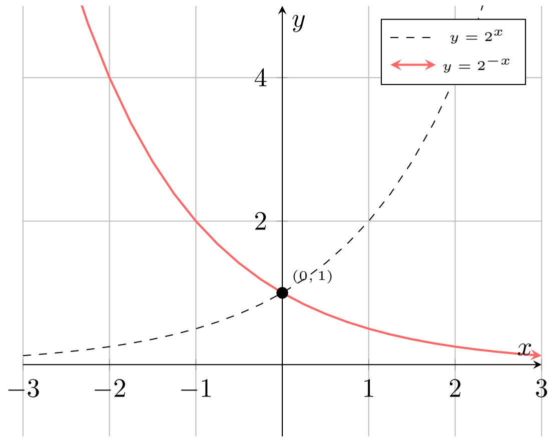
1. A **horizontal translation** of 1 unit to the **left**. This is because of the $(x + 1)$ term in the exponent.
2. A **vertical translation** of 4 units **upwards**. This is because of the $+4$ term.

The horizontal asymptote of $y = 2^x$ is $y = 0$. The vertical translation shifts the asymptote to $\boldsymbol{y = 4}$.



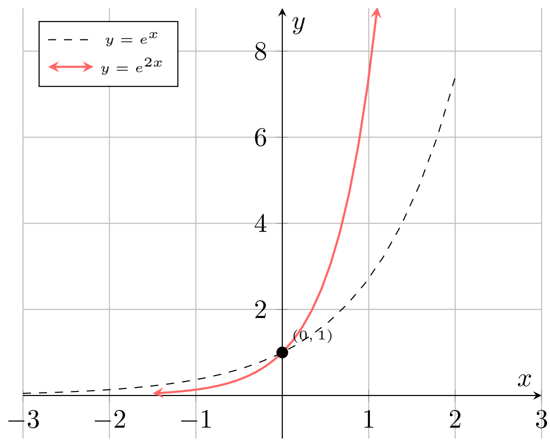
Ex 24: Describe the transformation that maps the graph of $y = 2^x$ to the graph of $f(x) = 2^{-x}$.

Answer: The function $f(x) = 2^{-x}$ is a transformation of the base function $y = 2^x$.
The transformation is a **reflection in the y-axis**.
This occurs because the input x is replaced with $-x$. The horizontal asymptote remains $y = 0$.



Ex 25: Describe the transformation that maps the graph of $y = e^x$ to the graph of $f(x) = e^{2x}$.

Answer: The function $f(x) = e^{2x}$ is a transformation of the base function $y = e^x$.
The transformation is a **horizontal stretch** by a factor of $\mathbf{\frac{1}{2}}$.
This means the graph is compressed horizontally towards the y-axis. The horizontal asymptote remains $y = 0$.

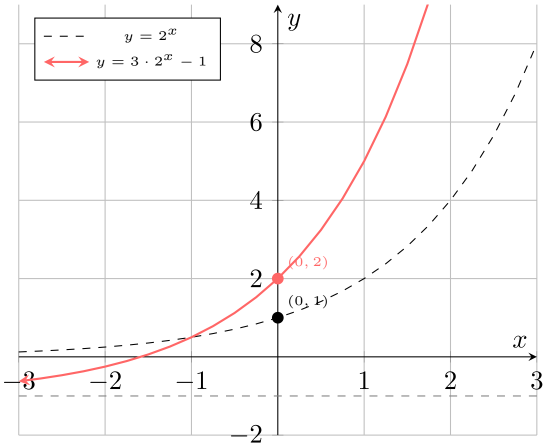


Ex 26: Describe the sequence of transformations that maps the graph of $y = 2^x$ to the graph of $f(x) = 3 \cdot 2^x - 1$.

Answer: The function $f(x) = 3 \cdot 2^x - 1$ is a transformation of the base function $y = 2^x$.
The transformations are:

1. A **vertical stretch** by a factor of **3**.
2. A **vertical translation** of 1 unit **downwards**.

The horizontal asymptote of $y = 2^x$ is $y = 0$. After the vertical translation, the new asymptote is $\boldsymbol{y = -1}$.



D EXPONENTIAL MODELS

D.1 MODELING REAL-WORLD SITUATIONS WITH EXPONENTIAL FUNCTIONS

Ex 27: A population of bacteria doubles every second. At time $x = 0$, there is a single bacterium.
Find the function to model this growth.

$$P(x) = \boxed{2^x}$$

Answer: Let $P(x)$ be the population of bacteria after x seconds. We have:

$$\begin{aligned} P(0) &= 1 = 2^0 \\ P(1) &= 2 = 2^1 \\ P(2) &= 4 = 2^2 \\ &\dots \\ P(x) &= 2^x \end{aligned}$$

So, the population after x seconds is $P(x) = 2^x$.

Ex 28: A species of bear is introduced to a large island off Alaska where previously there were no bears. 6 pairs of bears were introduced in 1998. It is expected that the population will increase according to $B(t) = B_0 \times (1.13)^t$ where t is the time, in years, since the introduction.

1. Find B_0.

$\boxed{12}$ bears

2. Find the expected bear population in 2018.

$\boxed{138}$ bears (round to the nearest integer)

3. Find the expected percentage increase in population from 1998 to 2018.

$\boxed{1050}$% (round to the nearest ten)

Answer:

1. $B_0 = 6$ pairs $= 12$ bears.
2. 2018 is 20 years after 1998, so $t = 20$.

$$\begin{aligned} B(20) &= 12 \times (1.13)^{20} \\ &\approx 12 \times 11.523 \\ &\approx 138.3 \\ &\approx 138 \text{ bears} \end{aligned}$$

3. The expected percentage increase is

$$\frac{138.3 - 12}{12} \times 100\% \approx 1050\%$$

Ex 29: Sarah buys a piece of artwork for \$1500 that is expected to appreciate (increase in value) by 8% each year.

1. Determine a model for A_n, the value of the artwork after n years.

$$A_n = \boxed{1500 \times (1.08)^n}$$

2. Is this an example of exponential growth?

$\boxed{Yes}$

3. Calculate the estimated value of the artwork in 6 years' time.

\$ $\boxed{2380}$ (round to the nearest integer)

Answer:

1. Initial value $A_0 = \$1500$, annual growth rate $r = 8\%$.
The model is:

$$\begin{aligned} A_0 &= 1500 \\ A_1 &= 1500 \times 1.08 \\ A_2 &= 1500 \times (1.08)^2 \\ A_3 &= 1500 \times (1.08)^3 \\ &\vdots \\ A_n &= 1500 \times (1.08)^n \end{aligned}$$

So, $A_n = 1500 \times (1.08)^n$.

2. Yes, this is an example of exponential growth because the value is multiplied by the same factor (1.08) each year.
3. Substitute $n = 6$:

$$\begin{aligned} A_6 &= 1500 \times (1.08)^6 \\ &\approx 1500 \times 1.586874 \\ &\approx 2380 \end{aligned}$$

The estimated value in 6 years is \$2 380 (rounded to the nearest integer).

Ex 30: Maxime has an Uncle Scrooge coin worth \$500. Each year, the coin's value increases by 20%.

1. Determine a model for C_n, the value of the coin after n years.

$$C_n = \boxed{500 \times (1.20)^n}$$

2. Is this an example of exponential growth?

$\boxed{Yes}$

3. Calculate the estimated value of the coin in 6 years' time.

\$ $\boxed{1493}$ (round to the nearest integer)

Answer:

1. Initial value $C_0 = \$500$, annual growth rate $r = 20\%$.
   The model is:
$$\begin{aligned} C_0 &= 500 \\ C_1 &= 500 \times 1.20 \\ C_2 &= 500 \times (1.20)^2 \\ C_3 &= 500 \times (1.20)^3 \\ &\vdots \\ C_n &= 500 \times (1.20)^n \end{aligned}$$
   So, $C_n = 500 \times (1.20)^n$.
2. Yes, this is an example of exponential growth because the value is multiplied by the same factor (1.20) each year.
3. Substitute $n = 6$:
$$\begin{aligned} C_6 &= 500 \times (1.20)^6 \\ &\approx 500 \times 2.985984 \\ &\approx 1493 \end{aligned}$$
   The estimated value in 6 years is \$1493 (rounded to the nearest integer).

Ex 31: A certain radioactive substance loses 12% of its mass each year. Initially, the sample weighs 200 g.

1. Determine a model for M_n, the mass (in grams) remaining after n years.
$$M_n = \boxed{200 \times (0.88)^n}$$
2. Is this an example of exponential decay?
   $\boxed{Yes}$
3. Calculate the mass remaining after 10 years.
   $\boxed{56}$ g (round to the nearest integer)

Answer:

1. Initial mass $M_0 = 200$ g, annual loss rate $= 12\%$.
   The decay factor is $R = 1 - 0.12 = 0.88$.
   The model is:
$$M_n = 200 \times (0.88)^n$$
2. Yes, this is an example of exponential decay because the mass is multiplied by the same constant factor (0.88) each year.
3. Substitute $n = 10$:
$$\begin{aligned} M_{10} &= 200 \times (0.88)^{10} \\ &\approx 200 \times 0.2785 \\ &\approx 55.7 \end{aligned}$$
   So, the mass remaining after 10 years is **56 g** (rounded to the nearest integer).

D.2 SOLVING PROBLEMS USING EXPONENTIAL MODELS

Ex 32: The temperature, T, in degrees Celsius (°C), of a cup of coffee t minutes after it is poured is modelled by the function:
$$T(t) = 22 + 70e^{-kt}$$
where k is a positive constant.

1. Find the initial temperature of the coffee.
2. The temperature of the coffee is 65°C after 5 minutes. Find the value of k.
3. Find the temperature of the coffee after 15 minutes.
4. Find the rate at which the temperature of the coffee is decreasing at $t = 10$ minutes.
5. State the temperature of the room, giving a reason for your answer.

Answer:

1. Initial temperature is at $t = 0$. $T(0) = 22 + 70e^0 = 22 + 70 = \mathbf{92°C}$.
2. We are given $T(5) = 65$.
$$\begin{aligned} 65 &= 22 + 70e^{-5k} \\ 43 &= 70e^{-5k} \\ \frac{43}{70} &= e^{-5k} \\ \ln\left(\frac{43}{70}\right) &= -5k \\ k &= -\frac{1}{5}\ln\left(\frac{43}{70}\right) \approx \mathbf{0.0974} \end{aligned}$$
3. We use the value of k found in part (b) and set $t = 15$.
$$\begin{aligned} T(15) &= 22 + 70e^{-0.0974 \times 15} \\ &\approx \mathbf{38.2°C} \end{aligned}$$
4. The rate of change is the derivative, $T'(t)$.
$$\begin{aligned} T'(t) &= 70 \cdot (-k)e^{-kt} \\ &= -70ke^{-kt} \end{aligned}$$
   At $t = 10$:
$$\begin{aligned} T'(10) &= -70(0.0974)e^{-0.0974 \times 10} \\ &\approx -2.57 \end{aligned}$$
   The temperature is decreasing at a rate of $\mathbf{2.57°C}$ per minute.
5. As $t \to \infty$, the term $e^{-kt} \to 0$.
   Therefore, $\lim_{t \to \infty} T(t) = 22 + 70(0) = 22$.
   The coffee will cool down to the temperature of its surroundings. The temperature of the room is $\mathbf{22°C}$. This is the horizontal asymptote of the function.

Ex 33: The concentration of a drug in a patient's bloodstream, C, in milligrams per litre (mg/L), is modelled by the function:

$$C(t) = 80(e^{-0.2t} - e^{-1.5t})$$

where t is the time in hours after the drug was administered.

1. Find the concentration of the drug in the bloodstream after 2 hours.
2. Use your graphing display calculator to find the maximum concentration of the drug and the time at which it occurs.
3. Find the rate of change of the drug's concentration at $t = 4$ hours. Interpret the meaning of your answer.
4. Determine the long-term concentration of the drug in the bloodstream, justifying your answer.

Answer:

1. Substitute $t = 2$ into the function:

$$\begin{aligned} C(2) &= 80(e^{-0.2\times 2} - e^{-1.5\times 2}) \\ &= 80(e^{-0.4} - e^{-3}) \\ &\approx \mathbf{49.6} \text{ mg/L} \end{aligned}$$

2. By using a graphing display calculator to find the maximum of the function $C(t)$: The maximum concentration is approximately **51** mg/L, which occurs at $\boldsymbol{t \approx 1.6}$ hours.

3. The rate of change is the derivative, $C'(t)$.

$$C'(t) = 80(-0.2e^{-0.2t} + 1.5e^{-1.5t})$$

At $t = 4$:

$$\begin{aligned} C'(4) &= 80(-0.2e^{-0.8} + 1.5e^{-6}) \\ &\approx -6.90 \end{aligned}$$

The concentration is decreasing at a rate of **6.90** mg/L per hour.

4. As $t \to \infty$, both $e^{-0.2t} \to 0$ and $e^{-1.5t} \to 0$.
Therefore, $\lim_{t\to\infty} C(t) = 80(0 - 0) = 0$.
The long-term concentration is **0** mg/L, meaning the drug is eliminated from the bloodstream.