DIFFERENTIAL EQUATIONS

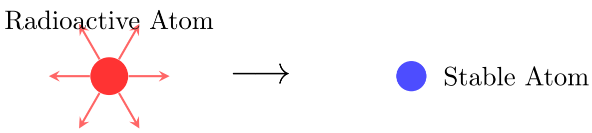
A FUNDAMENTALS OF DIFFERENTIAL EQUATIONS

A.1 MODELING WITH DIFFERENTIAL EQUATIONS

Ex 1: The rate at which a radioactive substance decays is proportional to the number of atoms, $N(t)$, remaining at time t. This is described by the first-order differential equation:

$$\frac{dN}{dt} = -kN$$

where k is the positive decay constant.



1. Let the initial number of atoms be N_0. State the initial condition for $N(t)$.
2. Verify that the general solution to this equation is $N(t) = Ae^{-kt}$, where A is an arbitrary constant.
3. Use the initial condition to find the particular solution for the number of atoms.

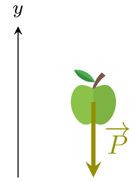
Answer:

1. **Initial Condition:** At time $t = 0$, the number of atoms is given as N_0. Therefore, the initial condition is $N(0) = N_0$.
2. **Verifying the General Solution:** We must show that $N(t) = Ae^{-kt}$ satisfies $\dfrac{dN}{dt} = -kN$.
$$\begin{aligned} \frac{dN(t)}{dt} &= \frac{d}{dt}\left(Ae^{-kt}\right) \\ &= -kAe^{-kt} \\ &= -kN(t) \end{aligned}$$
3. **Finding the Particular Solution:** We apply the initial condition $N(0) = N_0$ to the general solution:
$$\begin{aligned} N(0) &= Ae^{-k(0)} \\ N_0 &= A \cdot e^0 \\ A &= N_0 \end{aligned}$$
Substituting this value of A back into the general solution gives the well-known law of radioactive decay:
$$N(t) = N_0 e^{-kt}$$

Ex 2: An apple is dropped from rest at a height of 10 meters. Its vertical position, $y(t)$, is governed by the second-order differential equation:

$$\frac{d^2y}{dt^2} = -g$$

where g is the constant of gravitational acceleration.



1. State the initial conditions for position $y(0)$ and velocity $y'(0)$.
2. Verify that the general solution to this equation is $y(t) = -\frac{1}{2}gt^2 + At + B$.
3. Use the initial conditions to find the particular solution for the apple's motion.

Answer:

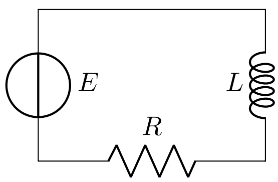
1. **Initial Conditions:**
   - The initial height is 10 meters, so $y(0) = 10$.
   - The apple is dropped "from rest," so its initial velocity is zero. Thus, $y'(0) = 0$.
2. **Verifying the General Solution:** We need to show that the second derivative of $y(t) = -\frac{1}{2}gt^2 + At + B$ is equal to $-g$.
   - First derivative (velocity): $y'(t) = \frac{d}{dt}\left(-\frac{1}{2}gt^2 + At + B\right) = -gt + A$.
   - Second derivative (acceleration): $y''(t) = \frac{d}{dt}(-gt + A) = -g$.

   The second derivative is indeed $-g$, so the general solution is correct.
3. **Finding the Particular Solution:** We apply the initial conditions to the general solution and its first derivative.
   - Using $y(0) = 10$:
   $$y(0) = -\frac{1}{2}g(0)^2 + A(0) + B \implies 10 = B$$
   - Using $y'(0) = 0$:
   $$y'(0) = -g(0) + A \implies 0 = A$$

   Substituting $A = 0$ and $B = 10$ into the general solution gives the particular solution:
   $$y(t) = -\frac{1}{2}gt^2 + 10$$

Ex 3: Consider a simple RL circuit with a resistor R, an inductor L, and a constant voltage source E. The current $I(t)$ in the circuit is governed by the first-order differential equation:

$$L\frac{dI}{dt} + RI = E$$

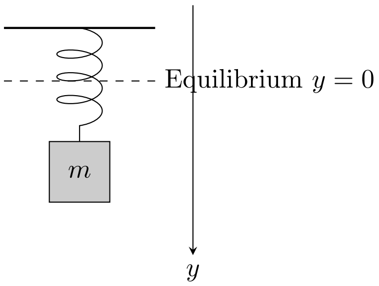


1. State the initial condition $I(0)$ if the circuit is switched on at time $t = 0$.
2. Verify that the general solution to this equation is $I(t) = \frac{E}{R} + Ae^{-\frac{R}{L}t}$, where A is an arbitrary constant.
3. Use the initial condition to find the particular solution for the current in the circuit.

Answer:

1. **Initial Condition:** Before the switch is closed at $t = 0$, no current flows. Due to the inductor, the current cannot change instantaneously. Therefore, the initial current is zero: $I(0) = 0$.
2. **Verifying the General Solution:** We must show that $I(t) = \frac{E}{R} + Ae^{-\frac{R}{L}t}$ satisfies $L\frac{dI}{dt} + RI = E$.
$$\begin{aligned} L\frac{dI(t)}{dt} + RI(t) &= L\frac{d}{dt}\left(\frac{E}{R} + Ae^{-\frac{R}{L}t}\right) + R\left(\frac{E}{R} + Ae^{-\frac{R}{L}t}\right) \\ &= L\left(-\frac{AR}{L}e^{-\frac{R}{L}t}\right) + E + ARe^{-\frac{R}{L}t} \\ &= -ARe^{-\frac{R}{L}t} + E + ARe^{-\frac{R}{L}t} \\ &= E \end{aligned}$$
3. **Finding the Particular Solution:** We apply the initial condition $I(0) = 0$ to the general solution:
$$I(0) = \frac{E}{R} + Ae^{-\frac{R}{L}(0)} \implies 0 = \frac{E}{R} + A \cdot 1$$
$$A = -\frac{E}{R}$$
Substituting this value of A back into the general solution gives the particular solution:
$$I(t) = \frac{E}{R} - \frac{E}{R}e^{-\frac{R}{L}t} = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

Ex 4: A mass m is attached to a vertical spring. According to Hooke's Law and Newton's Second Law, its displacement $y(t)$ from the equilibrium position is governed by the second-order differential equation:
$$m\frac{d^2y}{dt^2} = -ky$$
where k is the positive spring constant. This describes Simple Harmonic Motion.



1. The mass is pulled down to a position of $y = -A_0$ and released from rest at $t = 0$. State the initial conditions for $y(0)$ and $y'(0)$.
2. Verify that the general solution is $y(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$, where $\omega = \sqrt{\frac{k}{m}}$ and C_1, C_2 are arbitrary constants.
3. Use the initial conditions to find the particular solution for the mass's motion.

Answer:

1. **Initial Conditions:** The initial position is given as $-A_0$, so $y(0) = -A_0$. The mass is released "from rest," so its initial velocity is zero, thus $y'(0) = 0$.
2. **Verifying the General Solution:** We must show that $y(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$ satisfies $my'' = -ky$.
   - First derivative (velocity):
   $$y'(t) = -\omega C_1\sin(\omega t) + \omega C_2\cos(\omega t)$$
   - Second derivative (acceleration):
   $$\begin{aligned} y''(t) &= -\omega^2 C_1\cos(\omega t) - \omega^2 C_2\sin(\omega t) \\ &= -\omega^2(C_1\cos(\omega t) + C_2\sin(\omega t)) \\ &= -\omega^2 y(t) \end{aligned}$$
   - Substitute back into the differential equation, using $\omega^2 = k/m$:
   $$my'' = m(-\omega^2 y) = m\left(-\frac{k}{m}y\right) = -ky$$

   The equation holds, so the general solution is correct.
3. **Finding the Particular Solution:** We apply the initial conditions.
   - Using $y(0) = -A_0$:
   $$\begin{aligned} y(0) &= C_1\cos(0) + C_2\sin(0) \\ -A_0 &= C_1(1) + C_2(0) \implies C_1 = -A_0 \end{aligned}$$
   - Using $y'(0) = 0$:
   $$\begin{aligned} y'(0) &= -\omega C_1\sin(0) + \omega C_2\cos(0) \\ 0 &= -\omega C_1(0) + \omega C_2(1) \implies 0 = \omega C_2 \implies C_2 = 0 \end{aligned}$$

   Substituting $C_1 = -A_0$ and $C_2 = 0$ gives the particular solution:
   $$y(t) = -A_0\cos(\omega t)$$

B SLOPE FIELDS

B.1 SKETCHING SLOPE FIELDS

Ex 5: Consider the differential equation $\frac{dy}{dx} = x$.

1. On a set of axes, sketch the slope field for integer coordinates where $-2 \le x \le 2$ and $-1 \le y \le 1$.
2. On your sketch, draw the particular solution curve that passes through the point $(0, -1)$.

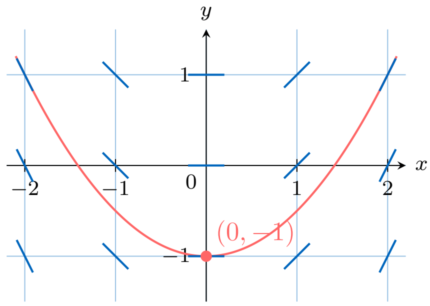
Answer:

1. **Sketching the field:** We calculate the slope $m = x$ at each integer x-coordinate. Notice that the slope does not depend on y, so all segments in the same vertical column will be parallel.

- At $x = -2$, the slope is $m = -2$.
- At $x = -1$, the slope is $m = -1$.
- At $x = 0$, the slope is $m = 0$ (horizontal segments).
- At $x = 1$, the slope is $m = 1$.
- At $x = 2$, the slope is $m = 2$.

We sketch these segments on the grid.

2. **Drawing the solution curve:** We start at the initial point $(0, -1)$ and draw a smooth curve that is tangent to the slope segments. We can see the curve is a parabola opening upwards.



Ex 6: Consider the differential equation $\dfrac{dy}{dx} = -y$.

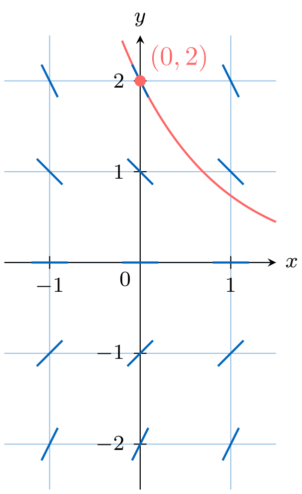
1. On a set of axes, sketch the slope field for integer coordinates where $-1 \leq x \leq 1$ and $-2 \leq y \leq 2$.
2. On your sketch, draw the particular solution curve that passes through the point $(0, 2)$.

Answer:

1. **Sketching the field:** We calculate the slope $m = -y$ at each integer y-coordinate. Notice that the slope does not depend on x, so all segments in the same horizontal row will be parallel.
   - At $y = 2$, the slope is $m = -2$.
   - At $y = 1$, the slope is $m = -1$.
   - At $y = 0$, the slope is $m = 0$ (horizontal segments along the x-axis).
   - At $y = -1$, the slope is $m = 1$.
   - At $y = -2$, the slope is $m = 2$.

   We sketch these segments on the grid.

2. **Drawing the solution curve:** We start at the initial point $(0, 2)$ and draw a smooth curve that is tangent to the slope segments. We can see the curve is an exponential decay function.



Ex 7: Consider the differential equation $\dfrac{dy}{dx} = x + y$.

1. On a set of axes, sketch the slope field for integer coordinates where $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$.
2. On your sketch, draw the particular solution curve that passes through the point $(0, 1)$.

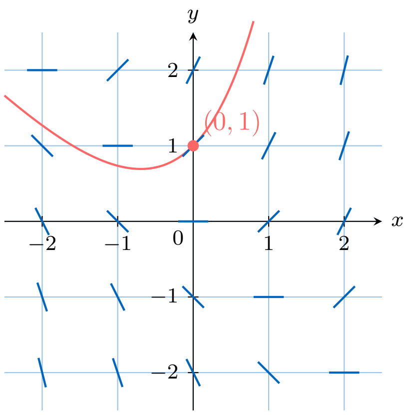
Answer:

1. **Sketching the field:** We calculate the slope $m = x + y$ at each integer point. The slope will be zero along the line $y = -x$.

y \ x	-2	-1	0	1	2
2	0	1	2	3	4
1	-1	0	1	2	3
0	-2	-1	0	1	2
-1	-3	-2	-1	0	1
-2	-4	-3	-2	-1	0

   We sketch these segments on the grid.

2. **Drawing the solution curve:** We start at the initial point $(0, 1)$ and draw a smooth curve that is tangent to the slope segments.



C SOLVING BY DIRECT INTEGRATION

C.1 SOLVING BY DIRECT INTEGRATION

Ex 8: Find the general solution to $\dfrac{dy}{dx} = 3x^2$.

$$y = \boxed{x^3 + C}$$

Answer: We rearrange and integrate both sides:

$$\begin{aligned}\frac{dy}{dx} &= 3x^2\\ dy &= 3x^2\,dx\\ \int dy &= \int 3x^2\,dx\\ y &= x^3 + C\end{aligned}$$

Ex 9: Find the general solution to $\dfrac{dy}{dx} = 2\cos(x) - 1$.

$$y = \boxed{2\sin(x) - x + C}$$

Answer: We rearrange and integrate both sides:

$$\begin{aligned}\frac{dy}{dx} &= 2\cos(x) - 1\\ dy &= (2\cos(x) - 1)\,dx\\ \int dy &= \int (2\cos(x) - 1)\,dx\\ y &= 2\sin(x) - x + C\end{aligned}$$

Ex 10: Find the general solution to $\frac{dy}{dx} = e^{2x}$.

$$y = \boxed{\frac{1}{2}e^{2x} + C}$$

Answer: We rearrange and integrate both sides:

$$\begin{aligned}\frac{dy}{dx} &= e^{2x}\\ dy &= e^{2x}\,dx\\ \int dy &= \int e^{2x}\,dx\\ y &= \frac{1}{2}e^{2x} + C\end{aligned}$$

Ex 11: Find the general solution to $\frac{dy}{dx} = \frac{2x}{1+x^2}$.

$$y = \boxed{\ln(1+x^2) + C}$$

Answer: We rearrange and integrate both sides:

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x}{1+x^2}\\ dy &= \frac{2x}{1+x^2}\,dx\\ \int dy &= \int \frac{2x}{1+x^2}\,dx\\ y &= \ln(1+x^2) + C\end{aligned}$$

(Note: The integral is of the form $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$. Since $1 + x^2 > 0$ for all x, the absolute value is not needed.)

Ex 12: Find the general solution to $\frac{dy}{dx} = \frac{x}{(x^2+2)^2}$.

$$y = \boxed{-\frac{1}{2(x^2+2)} + C}$$

Answer: We rearrange and integrate both sides. The integral on the right requires a substitution.

$$\begin{aligned}\frac{dy}{dx} &= \frac{x}{(x^2+2)^2}\\ dy &= \frac{x}{(x^2+2)^2}\,dx\\ \int dy &= \int \frac{x}{(x^2+2)^2}\,dx\end{aligned}$$

Let $u = x^2 + 2$. Then $\frac{du}{dx} = 2x$, which implies $x\,dx = \frac{1}{2}du$. Substituting this into the integral:

$$\begin{aligned}y &= \int \frac{1}{u^2}\cdot\frac{1}{2}\,du\\ &= \frac{1}{2}\int u^{-2}\,du\\ &= \frac{1}{2}\left(\frac{u^{-1}}{-1}\right) + C\\ &= -\frac{1}{2u} + C\\ &= -\frac{1}{2(x^2+2)} + C\end{aligned}$$

C.2 FINDING PARTICULAR SOLUTIONS BY INTEGRATION

Ex 13: Find the particular solution to $\frac{dy}{dx} = 3x^2$ that passes through the point $(1, 3)$.

$$y = \boxed{x^3 + 2}$$

Answer: First, we find the general solution by integration:

$$\begin{aligned}\frac{dy}{dx} &= 3x^2\\ y &= \int 3x^2\,dx = x^3 + C\end{aligned}$$

Now, we use the initial condition $y(1) = 3$ to find C:

$$\begin{aligned}3 &= (1)^3 + C\\ 3 &= 1 + C \implies C = 2\end{aligned}$$

The particular solution is $y = x^3 + 2$.

Ex 14: Find the particular solution to $\frac{dy}{dx} = 2\cos(x) - 1$ given the initial condition $y(\pi/2) = \pi$.

$$y = \boxed{2\sin(x) - x + \frac{3\pi}{2} - 2}$$

Answer: The general solution is $y = 2\sin(x) - x + C$. We use the initial condition $y(\pi/2) = \pi$:

$$\begin{aligned}\pi &= 2\sin(\pi/2) - \pi/2 + C\\ \pi &= 2(1) - \pi/2 + C\\ \pi &= 2 - \pi/2 + C\\ 3\pi/2 - 2 &= C\end{aligned}$$

The particular solution is $y = 2\sin(x) - x + \frac{3\pi}{2} - 2$.

Ex 15: Find the particular solution to $\frac{dy}{dx} = e^{2x}$ given that the solution curve passes through $(0, 5)$.

$$y = \boxed{\frac{1}{2}e^{2x} + \frac{9}{2}}$$

Answer: The general solution is $y = \frac{1}{2}e^{2x} + C$. We use the initial condition $y(0) = 5$:

$$\begin{aligned}5 &= \frac{1}{2}e^{2(0)} + C\\ 5 &= \frac{1}{2}(1) + C \implies C = \frac{9}{2}\end{aligned}$$

The particular solution is $y = \frac{1}{2}e^{2x} + \frac{9}{2}$.

Ex 16: Find the particular solution to $\dfrac{dy}{dx} = \dfrac{2x}{1+x^2}$ for which $y(0) = 3$.

$$y = \boxed{\ln(1+x^2)+3}$$

Answer: The general solution is $y = \ln(1+x^2) + C$. We use the initial condition $y(0) = 3$:

$$\begin{aligned} 3 &= \ln(1+0^2) + C \\ 3 &= \ln(1) + C \\ 3 &= 0 + C \implies C = 3 \end{aligned}$$

The particular solution is $y = \ln(1+x^2) + 3$.

Ex 17: Find the particular solution to $\dfrac{dy}{dx} = \dfrac{x}{(x^2+2)^2}$ given that $y(0) = -\frac{1}{2}$.

$$y = \boxed{-\frac{1}{2(x^2+2)} - \frac{1}{4}}$$

Answer: The general solution is $y = -\dfrac{1}{2(x^2+2)} + C$. We use the initial condition $y(0) = -1/2$:

$$\begin{aligned} -\frac{1}{2} &= -\frac{1}{2(0^2+2)} + C \\ -\frac{1}{2} &= -\frac{1}{4} + C \\ C &= -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4} \end{aligned}$$

The particular solution is $y = -\dfrac{1}{2(x^2+2)} - \dfrac{1}{4}$.

D SOLVING BY SEPARATION OF VARIABLES

D.1 SOLVING SEPARABLE EQUATIONS

Ex 18: Find the general solution to $\dfrac{dy}{dx} = \dfrac{1}{y}$.

$$y^2 = \boxed{2x} + C$$

Answer: We rearrange by separating the variables and then integrate both sides:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{y} \\ y\,dy &= dx \\ \int y\,dy &= \int dx \\ \frac{1}{2}y^2 &= x + C_1 \\ y^2 &= 2x + 2C_1 \end{aligned}$$

Letting $C = 2C_1$, the general solution is $y^2 = 2x + C$.

Ex 19: Find the general solution to $\dfrac{dy}{dx} = xy^3$.

$$y^{-2} = \boxed{-x^2} + C$$

Answer: We rearrange by separating the variables and then integrate both sides:

$$\begin{aligned} \frac{dy}{dx} &= xy^3 \\ \frac{1}{y^3}\,dy &= x\,dx \\ \int y^{-3}\,dy &= \int x\,dx \\ \frac{y^{-2}}{-2} &= \frac{x^2}{2} + C_1 \\ y^{-2} &= -x^2 - 2C_1 \end{aligned}$$

Letting $C = -2C_1$, the general solution is $y^{-2} = -x^2 + C$.

Ex 20: Find the general solution to $\dfrac{dy}{dx} = xe^{-y}$.

$$e^y = \boxed{\frac{1}{2}x^2} + C$$

Answer: We rearrange by separating the variables and then integrate both sides:

$$\begin{aligned} \frac{dy}{dx} &= \frac{x}{e^y} \\ e^y\,dy &= x\,dx \\ \int e^y\,dy &= \int x\,dx \\ e^y &= \frac{1}{2}x^2 + C \end{aligned}$$

This is the general solution in implicit form.

Ex 21: Find the general solution to $x^2\dfrac{dy}{dx} = y$.

$$y = C\boxed{e^{-1/x}}$$

Answer: We rearrange by separating the variables and then integrate both sides:

$$\begin{aligned} \frac{dy}{dx} &= \frac{y}{x^2} \\ \frac{1}{y}\,dy &= \frac{1}{x^2}\,dx \\ \int \frac{1}{y}\,dy &= \int x^{-2}\,dx \\ \ln|y| &= -x^{-1} + C' \\ |y| &= e^{-1/x + C'} \\ |y| &= e^{C'}e^{-1/x} \\ y &= Ce^{-1/x} \end{aligned}$$

D.2 FINDING PARTICULAR SOLUTIONS BY SEPARATION

Ex 22: Find the particular solution to $\dfrac{dy}{dx} = \dfrac{1}{y}$ that passes through the point $(4, 3)$.

$$y = \boxed{\sqrt{2x+1}}$$

Answer: First, we find the general solution by separating variables and integrating:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{y}\\ y\,dy &= dx\\ \int y\,dy &= \int dx\\ \frac{1}{2}y^2 &= x + C_1 \implies y^2 = 2x + C\end{aligned}$$

Now, we use the initial condition $y = 3$ when $x = 4$ to find C:

$$\begin{aligned}(3)^2 &= 2(4) + C\\ 9 &= 8 + C \implies C = 1\end{aligned}$$

The implicit particular solution is $y^2 = 2x + 1$. Since the initial condition specifies a positive y-value ($y = 3$), we take the positive square root to find the explicit solution:

$$y = \sqrt{2x+1}$$

Ex 23: Find the particular solution to $\frac{dy}{dx} = xy^3$ given the initial condition $y(0) = 1$.

$$y = \boxed{\frac{1}{\sqrt{1-x^2}}}$$

Answer: First, we find the general solution:

$$\begin{aligned}\frac{dy}{dx} &= xy^3\\ \int y^{-3}\,dy &= \int x\,dx\\ \frac{y^{-2}}{-2} &= \frac{x^2}{2} + C_1 \implies y^{-2} = -x^2 + C\end{aligned}$$

Now, we use the initial condition $y(0) = 1$ to find C:

$$(1)^{-2} = -(0)^2 + C \implies C = 1$$

The particular solution is $y^{-2} = -x^2 + 1$, which can be written as $\frac{1}{y^2} = 1 - x^2$, or $y = \frac{1}{\sqrt{1-x^2}}$ (taking the positive root since $y(0)$ is positive).

Ex 24: Find the particular solution to $\frac{dy}{dx} = xe^{-y}$ given that the solution curve passes through $(0, 0)$.

$$y = \boxed{\ln(\frac{1}{2}x^2 + 1)}$$

Answer: First, we find the general solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{x}{e^y}\\ \int e^y\,dy &= \int x\,dx\\ e^y &= \frac{1}{2}x^2 + C\end{aligned}$$

Now, we use the initial condition $y(0) = 0$ to find C:

$$e^0 = \frac{1}{2}(0)^2 + C \implies 1 = C$$

The implicit particular solution is $e^y = \frac{1}{2}x^2 + 1$. The explicit solution is $y = \ln\left(\frac{1}{2}x^2 + 1\right)$.

Ex 25: Find the particular solution to $x^2\frac{dy}{dx} = y$ for which $y(1) = 3$.

$$y = \boxed{3e^{1-1/x}}$$

Answer: First, we find the general solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x^2}\\ \int \frac{1}{y}\,dy &= \int x^{-2}\,dx\\ \ln|y| &= -x^{-1} + C_1\\ y &= Ae^{-1/x} \quad (\text{where } A = \pm e^{C_1})\end{aligned}$$

Now, we use the initial condition $y(1) = 3$ to find A:

$$\begin{aligned}3 &= Ae^{-1/1}\\ 3 &= Ae^{-1} \implies A = 3e\end{aligned}$$

Substituting this value of A gives the particular solution:

$$y = (3e)e^{-1/x} = 3e^{1-1/x}$$

E APPROXIMATING SOLUTIONS WITH EULER'S METHOD

E.1 APPLYING EULER'S METHOD

Ex 26: Consider the differential equation $\frac{dy}{dx} = y$ with the initial condition $y(0) = 1$.Using Euler's method with a step size of $h = 0.5$, find approximations for $y(0.5)$, $y(1.0)$, and $y(1.5)$.

- $y(0.5) \approx \boxed{1.5}$
- $y(1.0) \approx \boxed{2.25}$
- $y(1.5) \approx \boxed{3.375}$

Answer: We are given the initial point $(x_0, y_0) = (0, 1)$, a step size $h = 0.5$, and the function $f(x, y) = y$. We use the iterative formula

$$\begin{aligned}y_{n+1} &= y_n + h \cdot f(x_n, y_n)\\ y_{n+1} &= y_n + 0.5(y_n) = 1.5y_n\end{aligned}$$

- **Step 1: Find an approximation for y(0.5)**
  Here, $n = 0$. We use $(x_0, y_0) = (0, 1)$.

  $$\begin{aligned}y_1 &= y_0 + 0.5(y_0)\\ &= 1 + 0.5(1)\\ &= 1.5\end{aligned}$$

  Thus, $y(0.5) \approx 1.5$.

- **Step 2: Find an approximation for y(1.0)**
  Here, $n = 1$. We use the previous result, $(x_1, y_1) = (0.5, 1.5)$.

  $$\begin{aligned}y_2 &= y_1 + 0.5(y_1)\\ &= 1.5 + 0.5(1.5)\\ &= 1.5 + 0.75 = 2.25\end{aligned}$$

  Thus, $y(1.0) \approx 2.25$.

- **Step 3: Find an approximation for y(1.5)**
Here, $n = 2$. We use the previous result, $(x_2, y_2) = (1.0, 2.25)$.
$$\begin{aligned} y_3 &= y_2 + 0.5(y_2) \\ &= 2.25 + 0.5(2.25) \\ &= 2.25 + 1.125 = 3.375 \end{aligned}$$
Thus, $y(1.5) \approx 3.375$.

Ex 27: Consider the differential equation $\frac{dy}{dx} = x - y$ with the initial condition $y(0) = 1$.
Using Euler's method with a step size of $h = 0.5$, find approximations for $y(0.5)$, $y(1.0)$, and $y(1.5)$.

- $y(0.5) \approx \boxed{0.5}$
- $y(1.0) \approx \boxed{0.5}$
- $y(1.5) \approx \boxed{0.75}$

Answer: We are given the initial point $(x_0, y_0) = (0, 1)$, a step size $h = 0.5$, and the function $f(x, y) = x - y$. We use the iterative formula
$$\begin{aligned} y_{n+1} &= y_n + h \cdot f(x_n, y_n) \\ y_{n+1} &= y_n + 0.5(x_n - y_n) \end{aligned}$$

- **Step 1: Find an approximation for y(0.5)**
Here, $n = 0$. We use $(x_0, y_0) = (0, 1)$.
$$\begin{aligned} y_1 &= y_0 + 0.5(x_0 - y_0) \\ &= 1 + 0.5(0 - 1) \\ &= 0.5 \end{aligned}$$
Thus, $y(0.5) \approx 0.5$.
- **Step 2: Find an approximation for y(1.0)**
Here, $n = 1$. We use the previous result, $(x_1, y_1) = (0.5, 0.5)$.
$$\begin{aligned} y_2 &= y_1 + 0.5(x_1 - y_1) \\ &= 0.5 + 0.5(0.5 - 0.5) \\ &= 0.5 \end{aligned}$$
Thus, $y(1.0) \approx 0.5$.
- **Step 3: Find an approximation for y(1.5)**
Here, $n = 2$. We use the previous result, $(x_2, y_2) = (1.0, 0.5)$.
$$\begin{aligned} y_3 &= y_2 + 0.5(x_2 - y_2) \\ &= 0.5 + 0.5(1.0 - 0.5) \\ &= 0.75 \end{aligned}$$
Thus, $y(1.5) \approx 0.75$.

Ex 28: Consider the differential equation $\frac{dy}{dx} - y = yx^2$ with the initial condition $y(0) = 1$.Using Euler's method with a step size of $h = 0.2$, find approximations for $y(0.2)$, $y(0.4)$, and $y(0.6)$. Round your answers to four decimal places where necessary.

- $y(0.2) \approx \boxed{1.2}$
- $y(0.4) \approx \boxed{1.4496}$
- $y(0.6) \approx \boxed{1.7859}$

Answer: First, we rearrange the differential equation to isolate $\frac{dy}{dx}$:
$$\frac{dy}{dx} = y + yx^2 = y(1 + x^2)$$
We are given the initial point $(x_0, y_0) = (0, 1)$, a step size $h = 0.2$, and the function $f(x, y) = y(1 + x^2)$. We use the iterative formula:
$$\begin{aligned} y_{n+1} &= y_n + h \cdot f(x_n, y_n) \\ y_{n+1} &= y_n + 0.2 \cdot y_n(1 + x_n^2) \end{aligned}$$

- **Step 1: Find an approximation for y(0.2)**
Here, $n = 0$. We use $(x_0, y_0) = (0, 1)$.
$$\begin{aligned} y_1 &= y_0 + 0.2 \cdot y_0(1 + x_0^2) \\ &= 1 + 0.2 \cdot 1(1 + 0^2) \\ &= 1 + 0.2 = 1.2 \end{aligned}$$
Thus, $y(0.2) \approx 1.2$.
- **Step 2: Find an approximation for y(0.4)**
Here, $n = 1$. We use the previous result, $(x_1, y_1) = (0.2, 1.2)$.
$$\begin{aligned} y_2 &= y_1 + 0.2 \cdot y_1(1 + x_1^2) \\ &= 1.2 + 0.2 \cdot (1.2)(1 + 0.2^2) \\ &= 1.2 + 0.24(1.04) \\ &= 1.2 + 0.2496 = 1.4496 \end{aligned}$$
Thus, $y(0.4) \approx 1.4496$.
- **Step 3: Find an approximation for y(0.6)**
Here, $n = 2$. We use the previous result, $(x_2, y_2) = (0.4, 1.4496)$.
$$\begin{aligned} y_3 &= y_2 + 0.2 \cdot y_2(1 + x_2^2) \\ &= 1.4496 + 0.2 \cdot (1.4496)(1 + 0.4^2) \\ &= 1.4496 + 0.28992(1.16) \\ &= 1.4496 + 0.3363072 \approx 1.7859 \end{aligned}$$
Thus, $y(0.6) \approx 1.7859$.