DIFFERENTIAL EQUATIONS

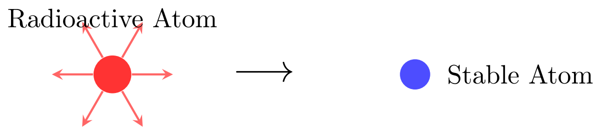
A FUNDAMENTALS OF DIFFERENTIAL EQUATIONS

A.1 MODELING WITH DIFFERENTIAL EQUATIONS

Ex 1: The rate at which a radioactive substance decays is proportional to the number of atoms, $N(t)$, remaining at time t. This is described by the first-order differential equation:

$$\frac{dN}{dt} = -kN$$

where k is the positive decay constant.

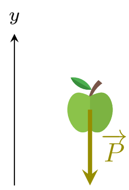


1. Let the initial number of atoms be N_0. State the initial condition for $N(t)$.
2. Verify that the general solution to this equation is $N(t) = Ae^{-kt}$, where A is an arbitrary constant.
3. Use the initial condition to find the particular solution for the number of atoms.

Ex 2: An apple is dropped from rest at a height of 10 meters. Its vertical position, $y(t)$, is governed by the second-order differential equation:

$$\frac{d^2y}{dt^2} = -g$$

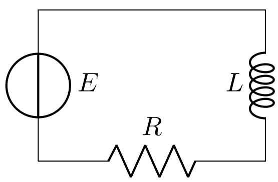
where g is the constant of gravitational acceleration.



1. State the initial conditions for position $y(0)$ and velocity $y'(0)$.
2. Verify that the general solution to this equation is $y(t) = -\frac{1}{2}gt^2 + At + B$.
3. Use the initial conditions to find the particular solution for the apple's motion.

Ex 3: Consider a simple RL circuit with a resistor R, an inductor L, and a constant voltage source E. The current $I(t)$ in the circuit is governed by the first-order differential equation:

$$L\frac{dI}{dt} + RI = E$$

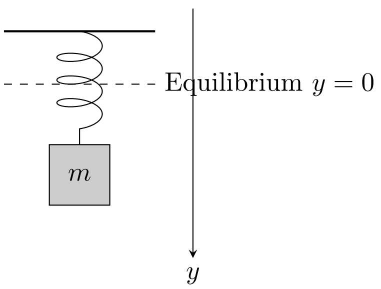


1. State the initial condition $I(0)$ if the circuit is switched on at time $t = 0$.
2. Verify that the general solution to this equation is $I(t) = \frac{E}{R} + Ae^{-\frac{R}{L}t}$, where A is an arbitrary constant.
3. Use the initial condition to find the particular solution for the current in the circuit.

Ex 4: A mass m is attached to a vertical spring. According to Hooke's Law and Newton's Second Law, its displacement $y(t)$ from the equilibrium position is governed by the second-order differential equation:

$$m\frac{d^2y}{dt^2} = -ky$$

where k is the positive spring constant. This describes Simple Harmonic Motion.



1. The mass is pulled down to a position of $y = -A_0$ and released from rest at $t = 0$. State the initial conditions for $y(0)$ and $y'(0)$.

2. Verify that the general solution is $y(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$, where $\omega = \sqrt{\frac{k}{m}}$ and C_1, C_2 are arbitrary constants.

3. Use the initial conditions to find the particular solution for the mass's motion.

B SLOPE FIELDS

B.1 SKETCHING SLOPE FIELDS

Ex 5: Consider the differential equation $\dfrac{dy}{dx} = x$.

1. On a set of axes, sketch the slope field for integer coordinates where $-2 \le x \le 2$ and $-1 \le y \le 1$.

2. On your sketch, draw the particular solution curve that passes through the point $(0, -1)$.

Ex 6: Consider the differential equation $\dfrac{dy}{dx} = -y$.

1. On a set of axes, sketch the slope field for integer coordinates where $-1 \le x \le 1$ and $-2 \le y \le 2$.

2. On your sketch, draw the particular solution curve that passes through the point $(0, 2)$.

Ex 7: Consider the differential equation $\frac{dy}{dx} = x + y$.

1. On a set of axes, sketch the slope field for integer coordinates where $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$.
2. On your sketch, draw the particular solution curve that passes through the point $(0, 1)$.

C SOLVING BY DIRECT INTEGRATION

C.1 SOLVING BY DIRECT INTEGRATION

Ex 8: Find the general solution to $\frac{dy}{dx} = 3x^2$.

$y =$

Ex 9: Find the general solution to $\frac{dy}{dx} = 2\cos(x) - 1$.

$y =$

Ex 10: Find the general solution to $\frac{dy}{dx} = e^{2x}$.

$y =$

Ex 11: Find the general solution to $\frac{dy}{dx} = \frac{2x}{1 + x^2}$.

$y =$

Ex 12: Find the general solution to $\frac{dy}{dx} = \frac{x}{(x^2 + 2)^2}$.

$y =$

C.2 FINDING PARTICULAR SOLUTIONS BY INTEGRATION

Ex 13: Find the particular solution to $\frac{dy}{dx} = 3x^2$ that passes through the point $(1, 3)$.

$y =$

Ex 14: Find the particular solution to $\frac{dy}{dx} = 2\cos(x) - 1$ given the initial condition $y(\pi/2) = \pi$.

$y =$

Ex 15: Find the particular solution to $\frac{dy}{dx} = e^{2x}$ given that the solution curve passes through $(0, 5)$.

$y =$

Ex 16: Find the particular solution to $\frac{dy}{dx} = \frac{2x}{1 + x^2}$ for which $y(0) = 3$.

$y =$

Ex 17: Find the particular solution to $\frac{dy}{dx} = \frac{x}{(x^2 + 2)^2}$ given that $y(0) = -\frac{1}{2}$.

$y =$

D SOLVING BY SEPARATION OF VARIABLES

D.1 SOLVING SEPARABLE EQUATIONS

Ex 18: Find the general solution to $\frac{dy}{dx} = \frac{1}{y}$.

$$y^2 = \square + C$$

Ex 19: Find the general solution to $\frac{dy}{dx} = xy^3$.

$$y^{-2} = \square + C$$

Ex 20: Find the general solution to $\frac{dy}{dx} = xe^{-y}$.

$$e^y = \square + C$$

Ex 21: Find the general solution to $x^2\frac{dy}{dx} = y$.

$$y = C\square$$

D.2 FINDING PARTICULAR SOLUTIONS BY SEPARATION

Ex 22: Find the particular solution to $\frac{dy}{dx} = \frac{1}{y}$ that passes through the point $(4, 3)$.

$$y = \square$$

Ex 23: Find the particular solution to $\frac{dy}{dx} = xy^3$ given the initial condition $y(0) = 1$.

$$y = \square$$

Ex 24: Find the particular solution to $\frac{dy}{dx} = xe^{-y}$ given that the solution curve passes through $(0, 0)$.

$$y = \square$$

Ex 25: Find the particular solution to $x^2\frac{dy}{dx} = y$ for which $y(1) = 3$.

$$y = \square$$

E APPROXIMATING SOLUTIONS WITH EULER'S METHOD

E.1 APPLYING EULER'S METHOD

Ex 26: Consider the differential equation $\frac{dy}{dx} = y$ with the initial condition $y(0) = 1$.Using Euler's method with a step size of $h = 0.5$, find approximations for $y(0.5)$, $y(1.0)$, and $y(1.5)$.

- $y(0.5) \approx \square$
- $y(1.0) \approx \square$
- $y(1.5) \approx \square$

Ex 27: Consider the differential equation $\frac{dy}{dx} = x - y$ with the initial condition $y(0) = 1$.
Using Euler's method with a step size of $h = 0.5$, find approximations for $y(0.5)$, $y(1.0)$, and $y(1.5)$.

- $y(0.5) \approx \square$
- $y(1.0) \approx \square$
- $y(1.5) \approx \square$

Ex 28: Consider the differential equation $\frac{dy}{dx} - y = yx^2$ with the initial condition $y(0) = 1$.Using Euler's method with a step size of $h = 0.2$, find approximations for $y(0.2)$, $y(0.4)$, and $y(0.6)$. Round your answers to four decimal places where necessary.

- $y(0.2) \approx \square$
- $y(0.4) \approx \square$
- $y(0.6) \approx \square$