DIFFERENTIAL CALCULUS

A DERIVATIVE

A.1 RATE OF CHANGE

A.1.1 FINDING RATE OF CHANGE

Ex 1: For the function $f(x) = x^2 + 1$, find the rate of change from $x = 1$ to $x = 2$.

Rate of change = $\square$

Ex 2: For the function $f(x) = (x - 1)(x + 1)$, find the rate of change from $x = -1$ to $x = 0$.

Rate of change = $\square$

Ex 3: For the function $f(x) = \sqrt{x}$, find the rate of change from $x = 1$ to $x = 4$.

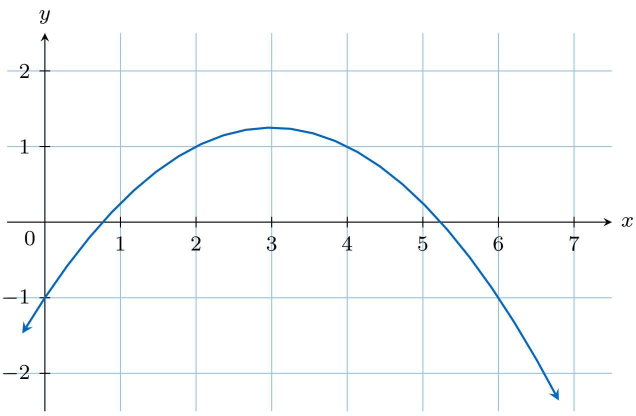
Rate of change = $\square$

Ex 4: For the function $f(x) = \ln(x)$, find the rate of change from $x = 1$ to $x = e$.

Rate of change = $\square$

A.1.2 FINDING RATE OF CHANGE FROM A GRAPH

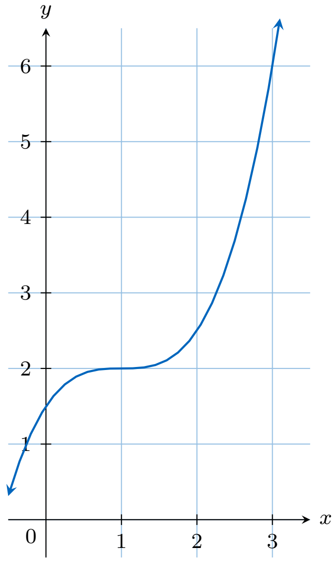
Ex 5: The graph of a function $y = f(x)$ is shown below.



Find the rate of change of the function from $x = 2$ to $x = 6$ graphically.

Rate of change = $\square$

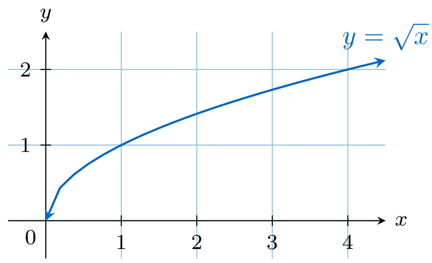
Ex 6: The graph of a function $y = f(x)$ is shown below.



Find the rate of change of the function from $x = 1$ to $x = 3$ graphically.

Rate of change = $\square$

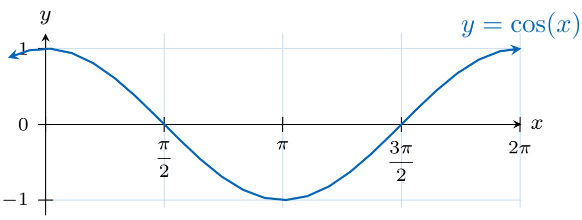
Ex 7: The graph of a function $y = f(x)$ is shown below.



Find the rate of change of the function from $x = 1$ to $x = 4$ graphically.

Rate of change = $\square$

Ex 8: The graph of a function $y = f(x)$ is shown below.



Find the rate of change of the function from $x = 0$ to $x = \pi$ graphically.

Rate of change = $\square$

A.1.3 MODELING WITH RATES OF CHANGE

Ex 9: An athlete completes a 100-meter race in 20 seconds. At the start of the race ($t = 0$ s), his distance from the starting line was 0 m. At the finish line ($t = 20$ s), his distance was 100 m. Calculate his average speed (the rate of change of distance with respect to time) over the course of the race.

Average speed = $\square$ m/s

Ex 10: A company's profit is recorded over a 5-year period. At the start of the period ($t = 0$ years), the profit was \$20,000. After 5 years ($t = 5$ years), the profit was \$80,000.
Calculate the average rate of change of profit (in dollars per year) over this period.

Average rate of change = ☐ \$/year

Ex 11: At 8:00 AM ($t = 0$ hours), the temperature in a room is 15°C. By noon ($t = 4$ hours), the temperature has risen to 25°C. Calculate the average rate of change of temperature (in degrees Celsius per hour) during this time.

Average rate of change = ☐ °C/hour

Ex 12: A biologist is monitoring a cell culture. At the start of the experiment ($t = 0$ days), the population is 500 cells. After 10 days, the population has grown to 4500 cells.
Calculate the average growth rate of the cell culture (in cells per day).

Average growth rate = ☐ cells/day

A.1.4 MODELING WITH RATES OF CHANGE

Ex 13: The temperature, T, of a cup of coffee is recorded at various times, t, after it is poured. The data is shown in the table below.

t (minutes)	0	2	5	9
T (°C)	90	75	60	50

1. Find the average rate of change of the temperature:
   (a) between $t = 0$ and $t = 2$ minutes.
   (b) between $t = 2$ and $t = 5$ minutes.
   (c) between $t = 5$ and $t = 9$ minutes.
2. What do these rates of change suggest about how the coffee is cooling?

Ex 14: The population of a town is recorded over several years. The data is shown in the table below.

t (year)	2000	2005	2015	2020
P (population)	5,000	5,500	8,000	12,000

1. Find the average rate of change of the population (in people per year):
   (a) between 2000 and 2005.
   (b) between 2005 and 2015.
   (c) between 2015 and 2020.
2. What do these rates of change suggest about the town's growth?

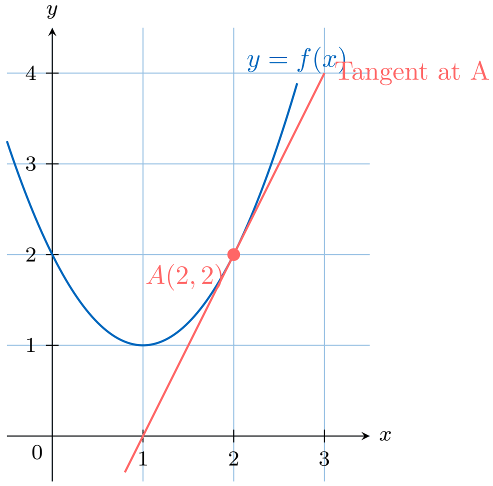
Ex 15: A swimmer completes a 400m freestyle race. Their split times are recorded every 100 meters, as shown in the table below.

t (seconds)	0	50	110	180	255
d (meters)	0	100	200	300	400

1. Find the swimmer's average speed (rate of change of distance with respect to time) for each 100m segment of the race:
   (a) from 0m to 100m.
   (b) from 100m to 200m.
   (c) from 200m to 300m.
   (d) from 300m to 400m.
2. What do these rates of change suggest about the swimmer's pacing during the race?

A.2.2 FINDING THE DERIVATIVE GRAPHICALLY

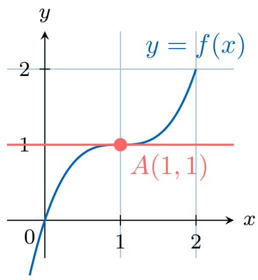
Ex 19: The graph of the function $f(x) = x^2 - 2x + 2$ and its tangent line at the point $A(2,2)$ are shown below.



Find the derivative of f at the point $x = 2$, i.e., $f'(2)$.

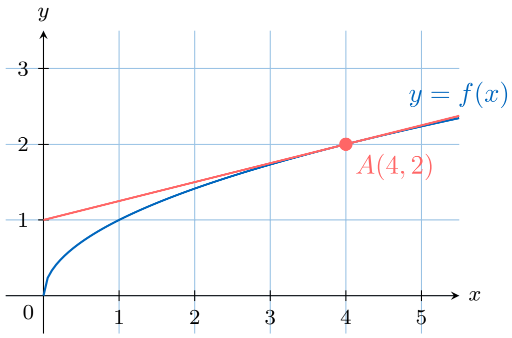
$$f'(2) =$$

Ex 20: The graph of $f(x) = (x-1)^3 + 1$ and its tangent at $A(1,1)$ are shown. Find $f'(1)$.



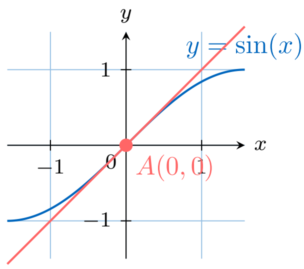
$$f'(1) =$$

Ex 21: The graph of $f(x) = \sqrt{x}$ and its tangent at $A(4,2)$ are shown. Find $f'(4)$.



$$f'(4) =$$

Ex 22: The graph of $f(x) = \sin(x)$ and its tangent at the origin $A(0,0)$ are shown. Find $f'(0)$.



A.2 LIMIT DEFINITION OF THE DERIVATIVE

A.2.1 CONJECTURING THE DERIVATIVE AT A POINT

Ex 16: For the function $f(x) = x^2$, find the rate of change from $x = 1$ to $x = 1 + h$:

- for $h = 1$:
- for $h = 0.1$:
- for $h = 0.01$:

Hence, conjecture the value of the derivative at $x = 1$.

$$f'(1) =$$

Ex 17: For the function $f(x) = x^3 + 1$, find the rate of change from $x = 0$ to $x = 0 + h$:

- for $h = 1$:
- for $h = 0.1$:
- for $h = 0.01$:

Hence, conjecture the value of the derivative at $x = 0$.

$$f'(0) =$$

Ex 18: For the function $f(x) = \sqrt{x}$, find the rate of change from $x = 1$ to $x = 1 + h$. Round your answers to 4 decimal places.

- for $h = 1$:
- for $h = 0.1$:
- for $h = 0.01$:

Hence, conjecture the value of the derivative at $x = 1$.

$$f'(1) =$$

$f'(0) = \square$

A.3 DERIVATIVE FUNCTION

A.3.1 FINDING THE DERIVATIVE FROM FIRST PRINCIPLES

Ex 23: For the function $f(x) = \frac{x}{2}$, find the derivative function $f'(x)$ using first principles.

Ex 24: For the function $f(x) = x^2$, find the derivative function $f'(x)$ using first principles.

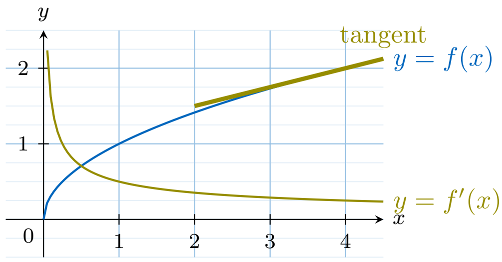
Ex 25: For the function $f(x) = \frac{1}{x}$, find the derivative function $f'(x)$ using first principles.

Ex 26: For the function $f(x) = 3$, find the derivative function $f'(x)$ using first principles.

Ex 27: For the function $f(x) = \sqrt{x}$, find the derivative function $f'(x)$ using first principles.

A.3.2 INTERPRETING THE GRAPH OF THE DERIVATIVE

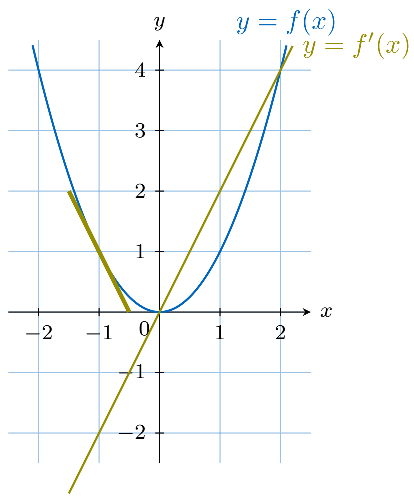
Ex 28: The graphs of a function $f(x)$ and its derivative function $f'(x)$ are shown below.



Use the graph of the derivative function to find the slope of the tangent to the graph of $f(x)$ at the point $x = 4$.

Slope at $x = 4$ is $\square$

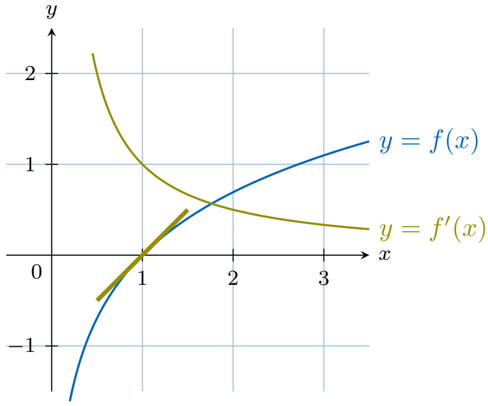
Ex 29: The graphs of a function $f(x)$ and its derivative function $f'(x)$ are shown below.



Use the graph of the derivative function to find the slope of the tangent to the graph of $f(x)$ at the point $x = -1$.

Slope at $x = -1$ is $\square$

Ex 30: The graphs of a function $f(x)$ and its derivative function $f'(x)$ are shown below.



Use the graph of the derivative function to find the slope of the tangent to the graph of $f(x)$ at the point $x = 1$.

Slope at $x = 1$ is $\square$

A.3.3 FINDING THE TANGENT SLOPE USING THE DERIVATIVE FUNCTION

Ex 31: The derivative of a function $f(x)$ is given by $f'(x) = 2x$. Find the slope of the tangent line to the graph of the original function, $y = f(x)$, at the point where $x = 1$.

Slope at $x = 1 = \square$

Ex 32: The derivative of a function $f(x)$ is given by $f'(x) = x^2 + 1$. Find the slope of the tangent line to the graph of the original function, $y = f(x)$, at the point where $x = 1$.

Slope at $x = 1 = \square$

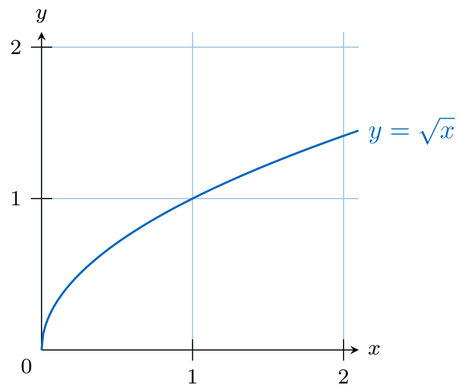
Ex 33: The derivative of a function $f(x)$ is given by $f'(x) = \cos(x)$. Find the slope of the tangent line to the graph of the original function, $y = f(x)$, at the point where $x = \frac{\pi}{2}$.

Slope at $x = \frac{\pi}{2} = \square$

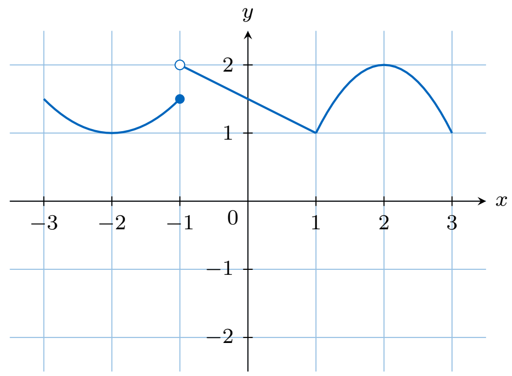
A.4 CONDITIONS OF DIFFERENTIABILITY

A.4.1 IDENTIFYING DIFFERENTIABILITY FROM A GRAPH

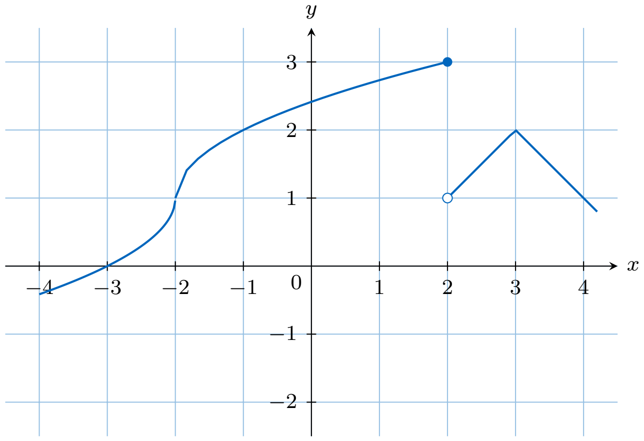
Ex 34: The graph of a function $y = f(x)$ is shown. State the x-values at which the function is not differentiable and give a reason for each.



Ex 35: The graph of a function $y = f(x)$ is shown. State the x-values at which the function is not differentiable and give a reason for each.



Ex 36: The graph of a function $y = f(x)$ is shown. State the x-values at which the function is not differentiable and give a reason for each.



B RULES OF DIFFERENTIATION

B.1 BASIC RULES AND POWER FUNCTIONS

B.1.1 PROVING BASIC RULES AND POWER FUNCTIONS

Ex 37: Prove that: if $f(x) = k$, then $f'(x) = 0$, where k is a constant.

Ex 38: Prove that: if $f(x) = x^2$, then $f'(x) = 2x$.

Ex 39: Prove that: if $f(x) = ku(x)$, then $f'(x) = ku'(x)$, where k is a constant.

Ex 40: Prove that: if $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$.

B.1.2 APPLYING THE POWER RULE

Ex 41: Find the derivative of $f(x) = x^4$.

$f'(x) =$ ☐

Ex 42: Find the derivative of $f(x) = x$.

$f'(x) =$ ☐

Ex 43: Find the derivative of $f(x) = \dfrac{1}{x^2}$.

$f'(x) =$ ☐

Ex 44: Find the derivative of $f(x) = \sqrt{x}$.

$f'(x) =$ ☐

Ex 45: Find the derivative of $f(x) = \dfrac{1}{x}$.

$f'(x) =$ ☐

B.1.3 DIFFERENTIATING POLYNOMIAL FUNCTIONS

Ex 46: Find the derivative of $f(x) = 3x - 2$.

$f'(x) =$ ☐

Ex 47: Find the derivative of $f(x) = x^2 + 4x - 5$.

$f'(x) =$ ☐

Ex 48: Find the derivative of $f(x) = 5x^3 - 2x^2 + 1$.

$f'(x) =$ ☐

Ex 49: Find the derivative of $f(x) = x^5 - \frac{1}{2}x^4 + 3x$.

$f'(x) =$ ☐

B.1.4 DIFFERENTIATING FUNCTIONS WITH FRACTIONAL AND NEGATIVE EXPONENTS

Ex 50: Find the derivative of $f(x) = 2x + 3\sqrt{x}$.

$f'(x) =$ ☐

Ex 51: Find the derivative of $f(x) = 3x + 2 + \frac{5}{x}$.

$f'(x) =$ ☐

Ex 52: Find the derivative of $f(x) = 3x\sqrt{x} - 2x$.

$f'(x) =$ ☐

Ex 53: Find the derivative of $f(x) = 2x^2 - \dfrac{1}{5x}$.

$f'(x) =$ ☐

B.1.5 EXPANDING BEFORE DIFFERENTIATING

Ex 54: By first simplifying the expression into a sum of terms, find the derivative of $f(x) = \dfrac{x-1}{x}$.

$f'(x) =$ ☐

Ex 55: By first expanding the expression, find the derivative of $f(x) = (x+1)^2$.

$f'(x) =$ ☐

Ex 56: By first expanding the expression, find the derivative of $f(x) = (2x^2 - 3)^2$.

$f'(x) =$ ☐

Ex 57: By first simplifying the expression, find the derivative of $f(x) = \dfrac{2x^3 - x}{x}$.

$f'(x) =$ ☐

B.2 CHAIN RULE

B.2.1 FORMING COMPOSITE FUNCTIONS

Ex 58: If $v(x) = x^3$ and $u(x) = 2x - 1$, find the composite function $f(x) = v(u(x))$.

$f(x) =$ ☐

Ex 59: If $v(x) = \dfrac{2}{x}$ and $u(x) = x^2 - 1$, find the composite function $f(x) = v(u(x))$.

$f(x) =$ ☐

Ex 60: If $v(x) = 3\sqrt{x}$ and $u(x) = x^4 + 1$, find the composite function $f(x) = v(u(x))$.

$f(x) =$ ☐

Ex 61: If $v(x) = 5x^2$ and $u(x) = 3x + 2$, find the composite function $f(x) = v(u(x))$.

$f(x) =$ ☐

B.2.2 DECOMPOSING COMPOSITE FUNCTIONS

Ex 62: Decompose the function $f(x) = (2x - 1)^3$ into an outer function v and an inner function u such that $f(x) = v(u(x))$.

$u(x) =$ ☐
$v(x) =$ ☐

Ex 63: Decompose the function $f(x) = \dfrac{2}{x^2 - 1}$ into an outer function v and an inner function u.

$u(x) =$ ☐
$v(x) =$ ☐

Ex 64: Decompose the function $f(x) = 3\sqrt{x^4 + 1}$ into an outer function v and an inner function u.

$u(x) =$ ☐
$v(x) =$ ☐

Ex 65: Decompose the function $f(x) = e^{2x-5}$ into an outer function v and an inner function u.

$u(x) =$ ☐
$v(x) =$ ☐

B.2.3 DIFFERENTIATING WITH THE CHAIN RULE

Ex 66: Find the derivative of $f(x) = \frac{1}{x^2+1}$.

$f'(x) =$ ☐

Ex 67: Find the derivative of $f(x) = 2\sqrt{2x - 1}$.

$f'(x) =$ ☐

Ex 68: Find the derivative of $f(x) = \sqrt[3]{x^3 + 8}$.

$f'(x) =$ ☐

Ex 69: Find the derivative of $f(x) = \dfrac{4}{\sqrt{x^2 + 9}}$.

$f'(x) =$ ☐

B.3 PRODUCT RULE

B.3.1 DIFFERENTIATING WITH THE PRODUCT RULE

Ex 70: Find the derivative of $f(x) = (x^2 - x)(x^3 + 2)$.

$f'(x) =$ ☐

Ex 71: Find the derivative of $f(x) = (x^2 + 3)(2x - 1)^4$.

$f'(x) =$ ☐

Ex 72: Find the derivative of $f(x) = \dfrac{1}{x^2}(x^3 + 1)$.

$f'(x) =$ ☐

Ex 73: Find the derivative of $f(x) = (3x + 1)\sqrt{x + 1}$.

$f'(x) =$ ☐

B.4 QUOTIENT RULE

B.4.1 DIFFERENTIATING WITH THE QUOTIENT RULE

Ex 74: Find the derivative of $f(x) = \frac{x^2-1}{x^2+1}$.

$f'(x) =$ ☐

Ex 75: Find the derivative of $f(x) = \frac{x^2}{x-1}$.

$f'(x) =$ ☐

Ex 76: Find the derivative of $f(x) = \frac{\sqrt{x}}{x+1}$.

$f'(x) =$ ☐

Ex 77: Find the derivative of $f(x) = \frac{x+2}{x^2-3}$.

$f'(x) =$ ☐

B.5 IMPLICIT DIFFERENTIATION

B.5.1 FINDING THE DERIVATIVE OF AN IMPLICIT FUNCTION

Ex 78: Find $\frac{dy}{dx}$ for the relation $x^3 + y^3 = 6$.

$\frac{dy}{dx} =$ ☐

Ex 79: Find $\frac{dy}{dx}$ for the relation $xy = 4$.

$\frac{dy}{dx} =$ ☐

Ex 80: Find $\frac{dy}{dx}$ for the relation $x^2 + 3xy - y^2 = 5$.

$\frac{dy}{dx} =$ ☐

B.5.2 FINDING THE SLOPE OF A TANGENT LINE OF AN IMPLICIT FUNCTION

Ex 81: Find the slope of the tangent to the ellipse $x^2 + 4y^2 = 8$ at the point $(2, 1)$.

Slope = ☐

Ex 82: Find the slope of the tangent to the curve $y^4 - x^3 = 1$ at the point $(2, \sqrt{3})$.

Slope = ☐

Ex 83: Find the slope of the tangent to the hyperbola $y^2 - x^2 = 3$ at the point $(1, 2)$.

Slope = ☐

C DERIVATIVES OF STANDARD FUNCTIONS

C.1 EXPONENTIAL FUNCTIONS

C.1.1 DIFFERENTIATING EXPONENTIAL FUNCTIONS: LEVEL 1

Ex 84: Find the derivative of $f(x) = e^{-x}$.

$f'(x) =$ ☐

Ex 85: Find the derivative of $f(x) = e^{x^2}$.

$f'(x) =$ ☐

Ex 86: Find the derivative of $f(x) = e^{x^2+2x+2}$.

$f'(x) =$ ☐

Ex 87: Find the derivative of $f(x) = e^{2\sqrt{x}}$.

$f'(x) =$ ☐

C.1.2 DIFFERENTIATING EXPONENTIAL FUNCTIONS: LEVEL 2

Ex 88: Find the derivative of $f(x) = e^x + e^{-x}$.

$f'(x) =$ ☐

Ex 89: Find the derivative of $f(x) = e^x(x^2 + 1)$.

$f'(x) =$ ☐

Ex 90: Find the derivative of $f(x) = \frac{x}{e^x}$.

$f'(x) =$ ☐

Ex 91: Find the derivative of $f(x) = \sqrt{e^x + 1}$.

$f'(x) =$ ☐

Ex 92: Find the derivative of $f(x) = (1 + e^x)^3$.

$f'(x) =$ ☐

C.2 LOGARITHMIC FUNCTIONS

C.2.1 DIFFERENTIATING LOGARITHMIC FUNCTIONS: LEVEL 1

Ex 93: Find the derivative of $f(x) = \ln(2x)$.

$f'(x) =$ ☐

Ex 94: Find the derivative of $f(x) = \ln(x^2 + 3)$.

$f'(x) =$ ☐

Ex 95: Find the derivative of $f(x) = \ln(x^2 + x + 1)$.

$f'(x) =$ ☐

C.2.2 DIFFERENTIATING LOGARITHMIC FUNCTIONS: LEVEL 2

Ex 96: Find the derivative of $f(x) = x^2 \ln(x)$.

$f'(x) =$ ☐

Ex 97: Find the derivative of $f(x) = \frac{\ln(x)}{x^2}$.

$f'(x) =$ ☐

Ex 98: Find the derivative of $f(x) = (\ln(x))^3$.

$f'(x) =$ ☐

Ex 99: Find the derivative of $f(x) = \ln(\ln(x))$.

$f'(x) =$ ☐

C.2.3 DIFFERENTIATING LOGARITHM FUNCTIONS OF THE FORM $\log_a(x)$

Ex 100: Find the derivative of $f(x) = \log_3(x)$.

$f'(x) =$ ☐

Ex 101: Find the derivative of $f(x) = \log_5(x^2 + 1)$.

$f'(x) =$ ☐

Ex 102: Find the derivative of $f(x) = x\log_{10}(x)$.

$f'(x) =$ ☐

Ex 103: Find the derivative of $f(x) = \dfrac{\log_7(x)}{x}$.

$f'(x) =$ ☐

C.3 TRIGONOMETRIC FUNCTIONS

C.3.1 DIFFERENTIATING TRIGONOMETRIC FUNCTIONS: LEVEL 1

Ex 104: Find the derivative of $f(x) = \sin(3x)$.

$f'(x) =$ ☐

Ex 105: Find the derivative of $f(x) = \cos(x^2)$.

$f'(x) =$ ☐

C.3.2 DIFFERENTIATING TRIGONOMETRIC FUNCTIONS: LEVEL 2

Ex 106: Find the derivative of $f(x) = x\cos(x)$.

$f'(x) =$ ☐

Ex 107: Find the derivative of $f(x) = \sin^3(x)$.

$f'(x) =$ ☐

Ex 108: Find the derivative of $f(x) = \frac{\sin(x)}{x}$.

$f'(x) =$ ☐

Ex 109: Find the derivative of $f(x) = e^x \sin(x)$.

$f'(x) =$ ☐

C.3.3 FINDING THE SLOPE OF A TANGENT LINE OF AN IMPLICIT FUNCTION

Ex 110: A curve is defined by the implicit equation $x^3 + \sin(y) = xy$.

1. Show that $\dfrac{dy}{dx} = \dfrac{y - 3x^2}{\cos(y) - x}$.
2. Find the slope of the tangent to the curve at the point $(0, \pi)$.

D SECOND DERIVATIVE

D.1 DEFINITION

D.1.1 CALCULATING THE FIRST AND SECOND DERIVATIVE: LEVEL 1

Ex 111: Find the first and second derivatives of $f(x) = x^4 - 3x^2 + 7$.

$f'(x) =$ ☐
$f''(x) =$ ☐

Ex 112: Find the first and second derivatives of $f(x) = e^{5x}$.

$f'(x) =$ ☐
$f''(x) =$ ☐

Ex 113: Find the first and second derivatives of $f(x) = \sin(2x)$.

$f'(x) =$ ☐
$f''(x) =$ ☐

D.1.2 CALCULATING THE FIRST AND SECOND DERIVATIVE: LEVEL 2

Ex 114: Find the first and second derivatives of $f(x) = x^2 \ln(x)$.

$f'(x) =$ ☐

$f''(x) =$ ☐

Ex 115: Find the first and second derivatives of $f(x) = \frac{x}{x+1}$.

$f'(x) =$ ☐

$f''(x) =$ ☐

Ex 116: Find the first and second derivatives of $f(x) = e^x \cos(x)$.

$f'(x) =$ ☐

$f''(x) =$ ☐