CURVES

A TANGENTS AND NORMALS

A.1 EQUATION OF THE TANGENT

A.1.1 FINDING THE EQUATION OF THE TANGENT

Ex 1: Find the equation of the tangent to $f(x) = x^2$ at $x = 1$.

$y =$

Ex 2: Find the equation of the tangent to $f(x) = x + \ln(x)$ at $x = 1$.

$y =$

Ex 3: Find the equation of the tangent to $f(x) = \sqrt{x^2+5}$ at $x = 2$.

$y =$

Ex 4: Find the equation of the tangent to $f(x) = \frac{1}{x+1}$ at $x = 1$.

$y =$

A.2 EQUATION OF THE NORMAL

A.2.1 FINDING THE EQUATION OF THE NORMAL

Ex 5: Find the equation of the normal to $f(x) = x^2$ at $x = 1$.

$y =$

Ex 6: Find the equation of the normal to $f(x) = x + \ln(x)$ at $x = 1$.

$y =$

Ex 7: Find the equation of the normal to $f(x) = \frac{e^x}{x^2+1}$ at $x = 1$.

$x =$

Ex 8: Find the equation of the normal to $f(x) = (x+1)\cos(x)$ at $x = 0$.

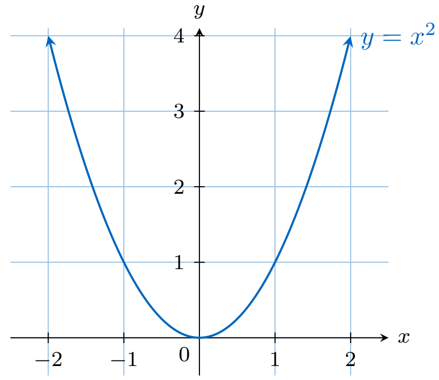
$y =$

B INCREASING AND DECREASING FUNCTIONS

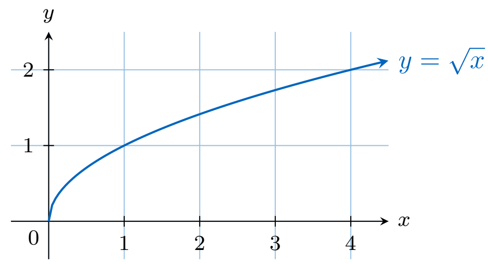
B.1 DEFINITION

B.1.1 DETERMINING VARIATIONS GRAPHICALLY

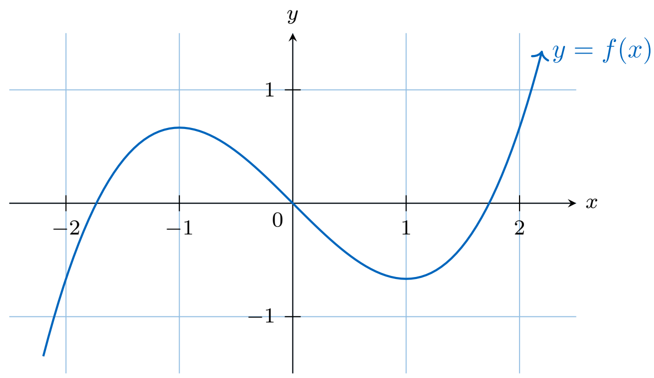
Ex 9: Graphically, find the variations for the function $f(x) = x^2$.



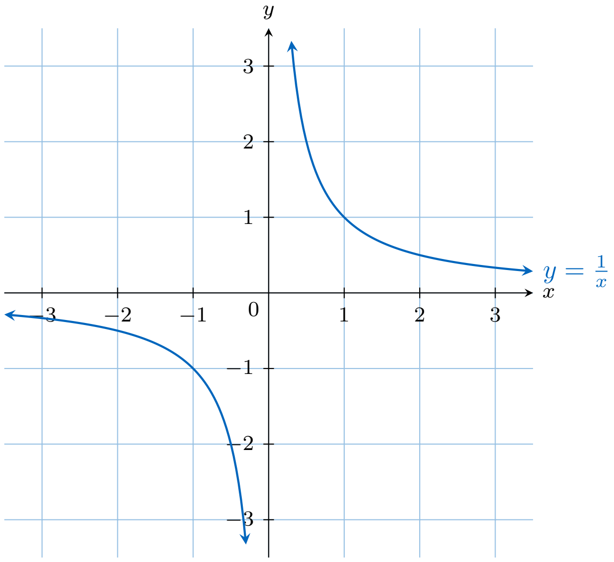
Ex 10: Graphically, find the variations for the function $f(x) = \sqrt{x}$.



Ex 11: Graphically, find the variations for the function $f(x) = \frac{x^3}{3} - x$.



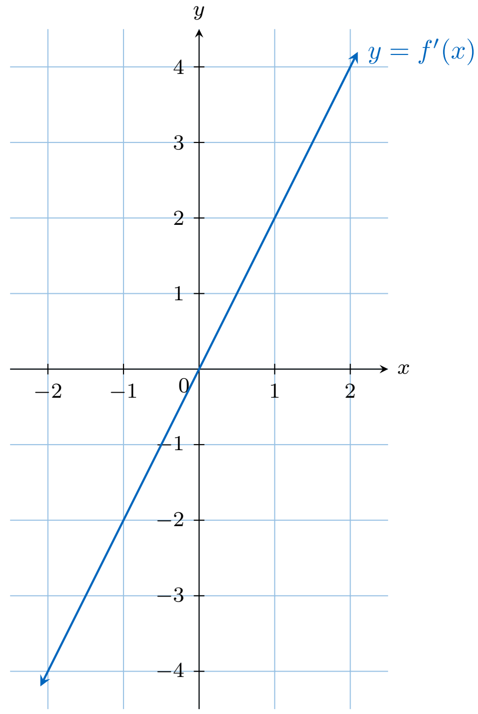
Ex 12: Graphically, find the variations for the function $f(x) = \frac{1}{x}$.



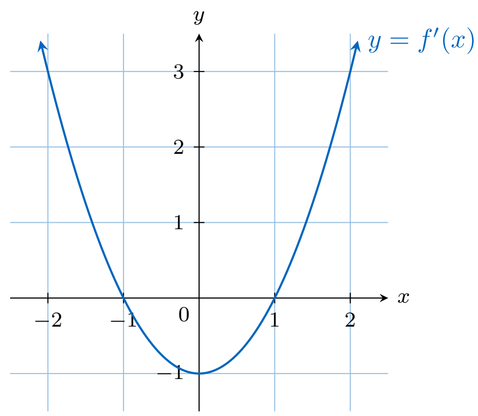
B.2 FIRST DERIVATIVE TEST

B.2.1 DETERMINING VARIATIONS FROM THE DERIVATIVE GRAPH

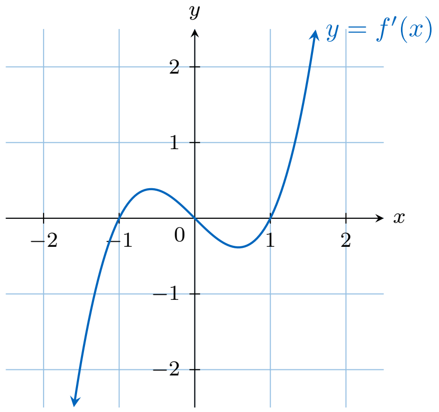
Ex 13: The graph of the derivative function, $f'(x) = 2x$, is shown below. Use it to determine the variations of the original function, f.



Ex 14: The graph of the derivative function, $f'(x) = x^2 - 1$, is shown below. Use it to determine the variations of the original function, f.



Ex 15: The graph of the derivative function, $f'(x) = x^3 - x$, is shown below. Use it to determine the variations of the original function, f.



B.2.2 STUDYING THE VARIATIONS OF STANDARD FUNCTIONS

Ex 16: Prove that $f(x) = \sqrt{x}$ is an increasing function on its domain.

Ex 17: Prove that $f(x) = \ln(x)$ is an increasing function on its domain.

B.2.3 STUDYING FUNCTION VARIATIONS

Ex 18: Find the variations of the function $f(x) = x^2$.

Ex 19: Find the variations of the function $f(x) = \frac{x^3}{3} - x$.

Ex 20: Find the variations of the function $f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x - 1$.

B.2.4 STUDYING FUNCTION VARIATIONS: LEVEL 2

Ex 21: Let $f(x) = \ln(x) - \frac{x^2}{2}$.

1. Show that $f'(x) = \frac{(1-x)(1+x)}{x}$.
2. Draw the sign diagram for $f'(x)$.

3. Hence, find the intervals where $y = f(x)$ is increasing or decreasing.

Ex 22: Let $f(x) = \dfrac{2-x}{x-1}$.

1. Show that $f'(x) = -\frac{1}{(x-1)^2}$.
2. Draw the sign diagram for $f'(x)$.
3. Hence, find the intervals where $y = f(x)$ is increasing or decreasing.

Ex 23: Let $f(x) = x + \dfrac{9}{x}$.

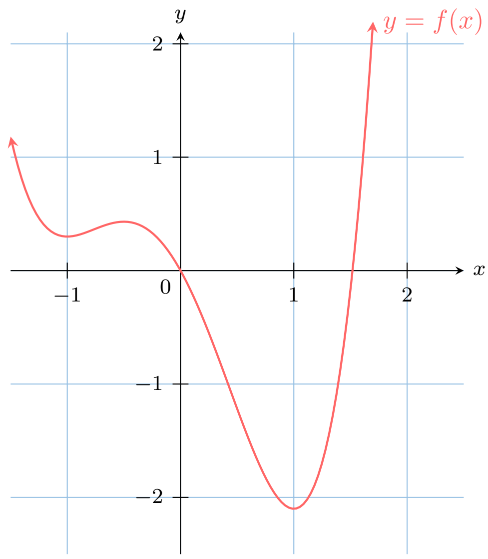
1. Show that $f'(x) = \dfrac{(x+3)(x-3)}{x^2}$.
2. Draw the sign diagram for $f'(x)$.
3. Hence, find the intervals where $y = f(x)$ is increasing or decreasing.

C EXTREMA OF FUNCTIONS

C.1 DEFINITIONS

C.1.1 IDENTIFYING EXTREMA FROM A GRAPH

MCQ 24: Consider the function f whose graph is shown below. Which of the following statements is true?

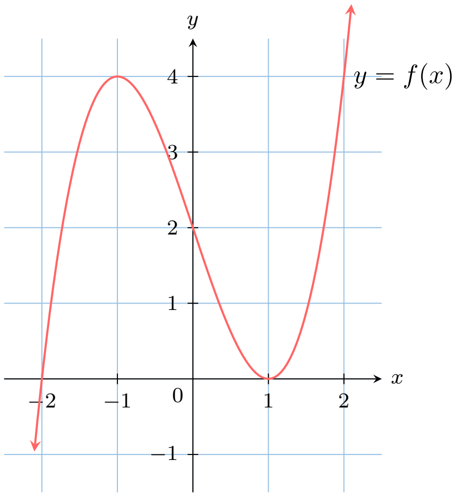


☐ The function has a global minimum at $x = -1$ and a local minimum at $x = 1$.

☐ The function has global minima at $x = -1$ and $x = 1$.

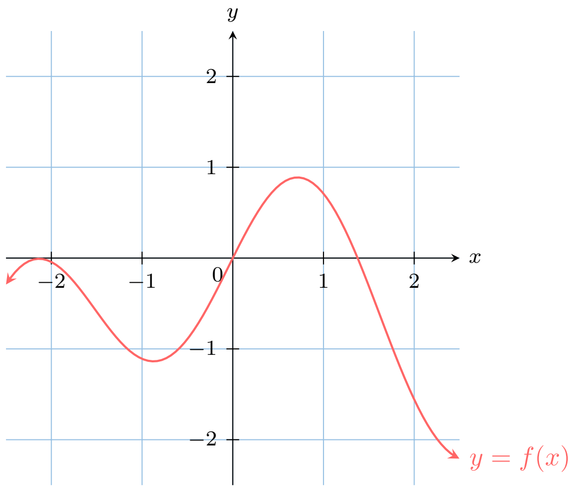
☐ The function has a local minimum at $x = -1$ and a global minimum at $x = 1$.

MCQ 25: Consider the function f whose graph is shown below. Which of the following statements is true?



- ☐ The function has a global maximum at $x = -1$ and a local minimum at $x = 1$.
- ☐ The function has a local maximum at $x = -1$ and a local minimum at $x = 1$.
- ☐ The function has a global maximum at $x = -1$ and a global minimum at $x = 1$.

MCQ 26: Consider the function f whose graph is shown below. Which of the following statements is true?



- ☐ The function has a global maximum at $x \approx 0.7$ and a local maximum at $x \approx -2.5$.
- ☐ The function has a local maximum at $x \approx 0.7$ and no global maximum.
- ☐ The function has a local maximum at $x \approx 0.7$ and a global maximum at $x \approx -2.5$.

C.2 FIRST DERIVATIVE TEST FOR LOCAL EXTREMA

C.2.1 FINDING AND CLASSIFYING EXTREMA: LEVEL 1

Ex 27: Let $f(x) = x^2 - 4x + 3$.

1. Find the derivative, $f'(x)$.
2. Find the x-coordinate of the stationary point of the function.
3. Hence, classify the stationary point as a local maximum or a local minimum.

Ex 28: Let $f(x) = -x^2 - 2x + 8$.

1. Find the derivative, $f'(x)$.
2. Find the x-coordinate of the stationary point of the function.
3. Hence, classify the stationary point as a local maximum or a local minimum.

Ex 29: Let $f(x) = 2x^3 - 3x^2 - 12x + 5$.

1. Find the derivative, $f'(x)$.
2. Find the x-coordinates of the stationary points of the function.
3. Hence, classify each stationary point as a local maximum or a local minimum.

C.2.2 FINDING AND CLASSIFYING EXTREMA: LEVEL 2

Ex 30: Let $f(x) = x\sqrt{4-x}$ for $x \leq 4$.

1. Show that the derivative is $f'(x) = \dfrac{8-3x}{2\sqrt{4-x}}$.
2. Find the coordinates of the stationary point on the graph of $y = f(x)$.
3. Using the first derivative test, determine the nature of this stationary point.
4. Find the global maximum and global minimum values of the function on the interval $[-5, 4]$.

Ex 31: Let $f(x) = \frac{\ln x}{x}$ for $x > 0$.

1. Show that the derivative is $f'(x) = \dfrac{1-\ln x}{x^2}$.
2. Find the exact coordinates of the stationary point on the graph of $y = f(x)$.
3. Using the first derivative test, determine the nature of this stationary point.
4. Find the global maximum and global minimum values of the function on the interval $[1, 4]$.

Ex 32: Let $f(x) = xe^{-x}$.

1. Show that the derivative is $f'(x) = \dfrac{1-x}{e^x}$.

2. Find the coordinates of the stationary point on the graph of $y = f(x)$.

3. Using the first derivative test, determine the nature of this stationary point.

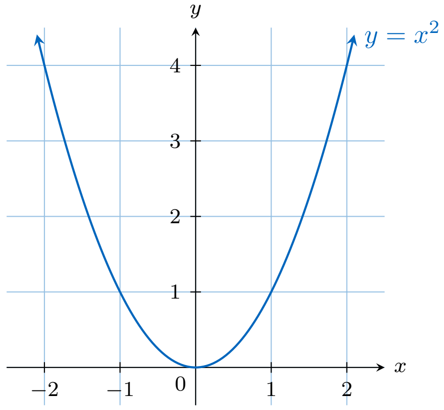
4. Find the global maximum and global minimum values of the function on the interval $[-1, 3]$.

D CONCAVITY

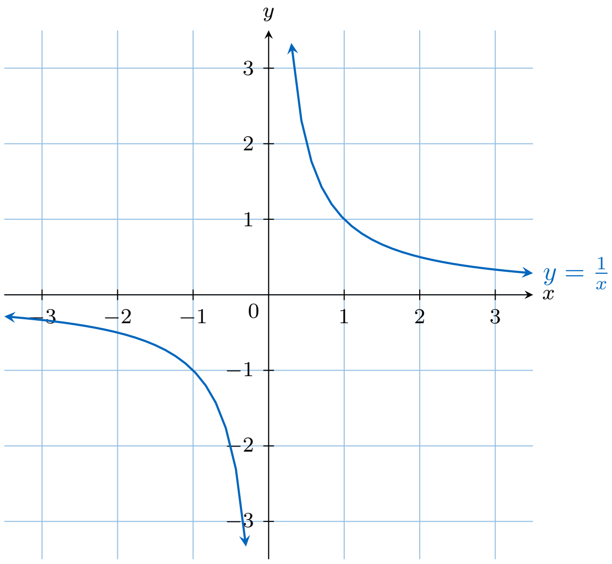
D.1 DEFINITION

D.1.1 DETERMINING CONCAVITY GRAPHICALLY

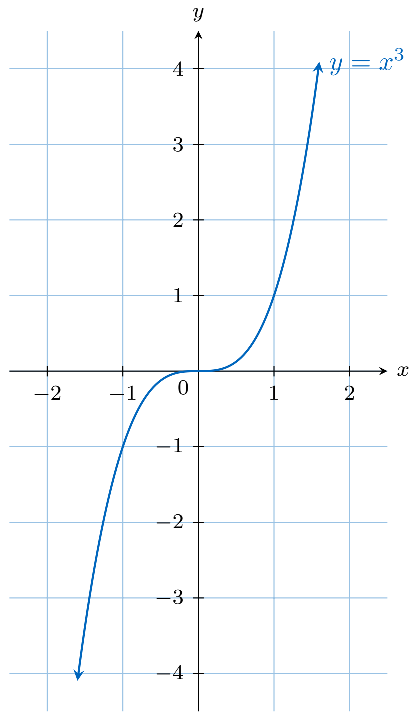
Ex 33: Graphically, determine the concavity of the function $f(x) = x^2$.



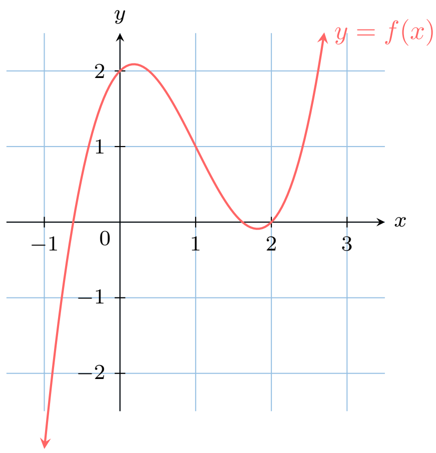
Ex 34: Graphically, determine the intervals of concavity for the function $f(x) = \frac{1}{x}$.



Ex 35: Graphically, determine the concavity of the function $f(x) = x^3$.



Ex 36: Graphically, find the point of inflection and describe the concavity for the function $f(x)$ shown below.



D.2 SECOND DERIVATIVE TEST FOR CONCAVITY

D.2.1 DETERMINING CONCAVITY: LEVEL 1

Ex 37: Let $f(x) = x^3$.

1. Find the second derivative, $f''(x)$.
2. Create a sign diagram for $f''(x)$.
3. Hence, determine the intervals where the function is concave up and concave down.

Ex 38: Let $f(x) = \frac{1}{x}$.

1. Find the second derivative, $f''(x)$.
2. Create a sign diagram for $f''(x)$.
3. Hence, determine the intervals where the function is concave up and concave down.

Ex 39: Let $f(x) = x^3 - 3x^2 + x$.

1. Find the second derivative, $f''(x)$.
2. Create a sign diagram for $f''(x)$.
3. Hence, determine the intervals where the function is concave up and concave down.

D.2.2 DETERMINING CONCAVITY: LEVEL 2

Ex 40: Let $f(x) = 2x^4 - 8x^3 + 12x^2 + 3$.

1. Show that $f''(x) = 24(x-1)^2$.
2. Hence, determine the concavity of the graph of $y = f(x)$.

Ex 41: The function f is defined by $f(x) = e^x \cos(x)$ for $x \in [0, 2\pi]$.

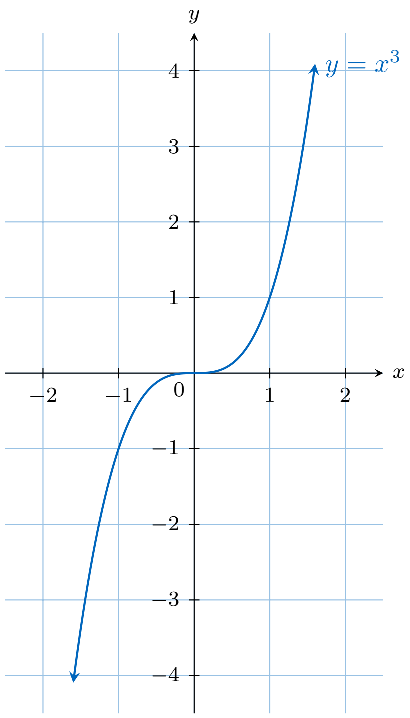
1. Find an expression for $f'(x)$.
2. Show that $f''(x) = -2e^x \sin(x)$.
3. Hence, find the interval(s) where the graph of f is concave down.

E POINTS OF INFLECTION

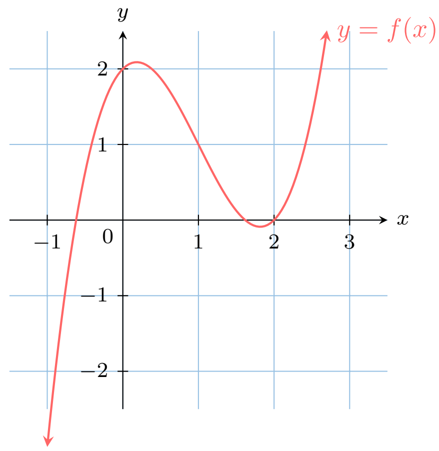
E.1 DEFINITION

E.1.1 IDENTIFYING POINTS OF INFLECTION FROM A GRAPH

Ex 42: Graphically, find the point of inflection for the function $f(x) = x^3$.



Ex 43: Graphically, find the point of inflection for the function $f(x)$ shown below.



E.2 SECOND DERIVATIVE TEST FOR POINTS OF INFLECTION

E.2.1 DETERMINING POINTS OF INFLECTION: LEVEL 1

Ex 44: Let $f(x) = x^3$.

1. Find the second derivative, $f''(x)$.
2. Find the x-coordinate of the potential point of inflection by solving $f''(x) = 0$.
3. Use a sign diagram for $f''(x)$ to show that a point of inflection exists at this x-coordinate.
4. Find the coordinates of the point of inflection and classify it as stationary or non-stationary.

Ex 45: Let $f(x) = x^3 - 3x^2 + x + 2$.

1. Find the second derivative, $f''(x)$.
2. Find the x-coordinate of the potential point of inflection.
3. Use a sign diagram for $f''(x)$ to show that a point of inflection exists at this x-coordinate.
4. Find the coordinates of the point of inflection and classify it as stationary or non-stationary.

Ex 46: Let $f(x) = \frac{1}{12}x^4 - \frac{1}{2}x^3 + x^2$.

1. Find the first and second derivatives of $f(x)$.
2. Find the x-coordinates of the potential points of inflection.
3. Use a sign diagram for $f''(x)$ to show that points of inflection exist at these x-coordinates.
4. Find the coordinates of the points of inflection and classify them as stationary or non-stationary.

E.2.2 DETERMINING POINTS OF INFLECTION: LEVEL 2

Ex 47: Let $f(x) = x^3 - 6x^2 + 12x - 5$.

1. Find expressions for $f'(x)$ and $f''(x)$.
2. Find the coordinates of the stationary point of $f(x)$.
3. Find the coordinates of the point of inflection.
4. Show that the stationary point is also the point of inflection.

Ex 48: Let $f(x) = xe^{-x}$.

1. Find expressions for $f'(x)$ and $f''(x)$.
2. Find the coordinates of the stationary point and determine its nature.

3. Find the coordinates of the point of inflection.
4. Find the interval(s) where the graph of f is concave down.