COUPLED DIFFERENTIAL EQUATIONS

A DEFINITIONS AND EQUILIBRIUM

A.1 CHECKING A SOLUTION

Ex 1: Consider the following system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = x + 2y \\ \dfrac{dy}{dt} = 2x + y \end{cases}$$

Show that the functions defined by $x(t) = e^{3t}$ and $y(t) = e^{3t}$ form a valid solution to this system.

Ex 2: Consider the following system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x \end{cases}$$

Show that the functions defined by $x(t) = \sin(t)$ and $y(t) = \cos(t)$ form a valid solution to this system.

Ex 3: Consider the following system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = y^2 \\ \dfrac{dy}{dt} = x \end{cases}$$

Show that the functions defined by $x(t) = -\dfrac{12}{t^3}$ and $y(t) = \dfrac{6}{t^2}$ form a valid solution to this system for $t \neq 0$.

A.2 IDENTIFYING COUPLED SYSTEMS

MCQ 4: Which of the following systems of differential equations is **coupled**?

- ☐ $\dfrac{dx}{dt} = 3x$
- ☐ $\begin{cases} \dfrac{dx}{dt} = x + y \\ \dfrac{dy}{dt} = x - y \end{cases}$
- ☐ $\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = y \end{cases}$

MCQ 5: Which of the following systems of differential equations is **coupled**?

- ☐ $\begin{cases} \dfrac{dx}{dt} = 2x \\ \dfrac{dy}{dt} = 5y \end{cases}$
- ☐ $\dfrac{dy}{dt} = 3y^2 - t$
- ☐ $\begin{cases} \dfrac{dx}{dt} = 3x - 2y \\ \dfrac{dy}{dt} = x + 4y \end{cases}$

A.3 DETERMINING THE EQUILIBRIUM POINT(S)

Ex 6: Consider the following system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = 2x - y \\ \dfrac{dy}{dt} = x + 2y \end{cases}$$

Determine the equilibrium point(s) of the system.

Ex 7: Consider the following system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = x - 1 \\ \dfrac{dy}{dt} = x + y \end{cases}$$

Determine the equilibrium point(s) of the system.

Ex 8: Consider the following system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = x^2 - 1 \\ \dfrac{dy}{dt} = x + y \end{cases}$$

Determine the equilibrium point(s) of the system.

B PHASE PORTRAIT

B.1 CALCULATING VELOCITY VECTORS

Ex 9: Consider the following system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = 2x - y \\ \dfrac{dy}{dt} = x + 2y \end{cases}$$

1. Find the velocity vector $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ at the point $(1, 1)$.

2. Find the velocity vector at the point $(-1, 1)$.

Ex 10: Consider the following system of non-linear coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = -x^2 + y \\ \dfrac{dy}{dt} = -(x - y)^2 \end{cases}$$

1. Find the velocity vector $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ at the point $(2, 5)$.

2. Find the velocity vector at the point $(1, 0)$.

Ex 11: Consider the following system involving products of variables:

$$\begin{cases} \dfrac{dx}{dt} = xy + 1 \\ \dfrac{dy}{dt} = y^2 - 3x \end{cases}$$

1. Find the velocity vector $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ at the point $(1, 2)$.

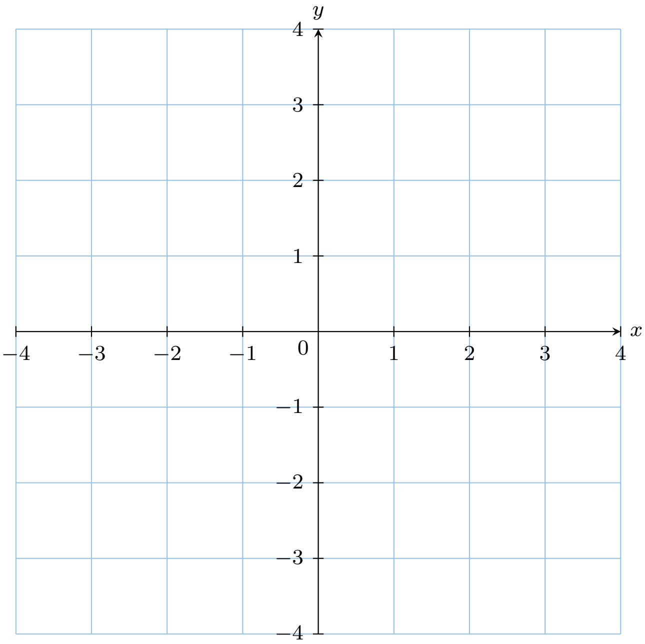
2. Find the velocity vector at the point $(2, -1)$.

B.2 SKETCHING PHASE PORTRAITS

Ex 12: Consider the following system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = \frac{x-y}{2} \\ \dfrac{dy}{dt} = \frac{x+y}{2} \end{cases}$$

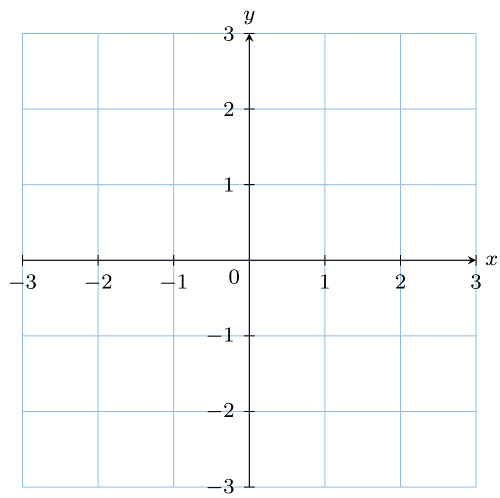
Sketch the phase portrait for the system by drawing the velocity vector at each grid point (x, y) where $x, y \in \{-2, -1, 0, 1, 2\}$.



Ex 13: Consider the following system of coupled differential equations:

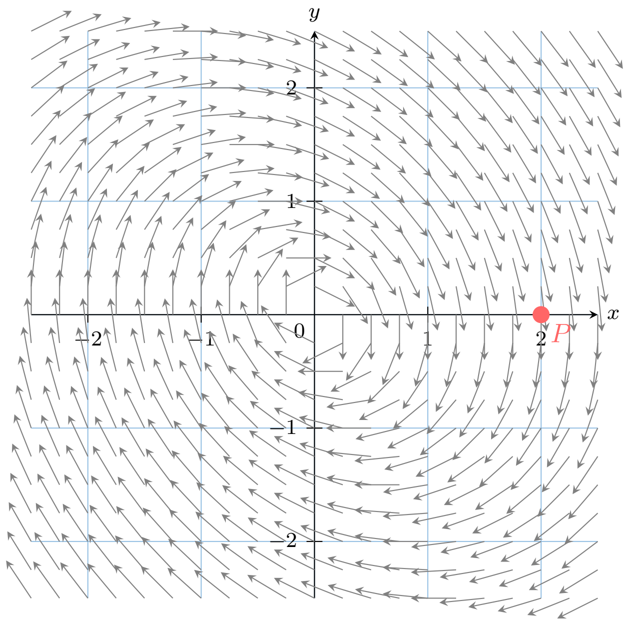
$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = x \end{cases}$$

Sketch the phase portrait for the system by drawing the velocity vector at each grid point (x, y) where $x, y \in \{-2, -1, 0, 1, 2\}$. The vectors should be scaled by a factor of 0.5 for clarity.

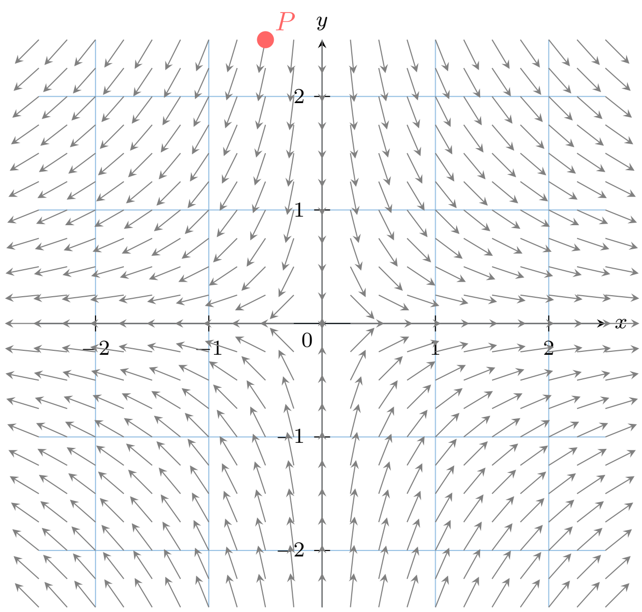


B.3 SKETCHING TRAJECTORIES FROM PHASE PORTRAITS

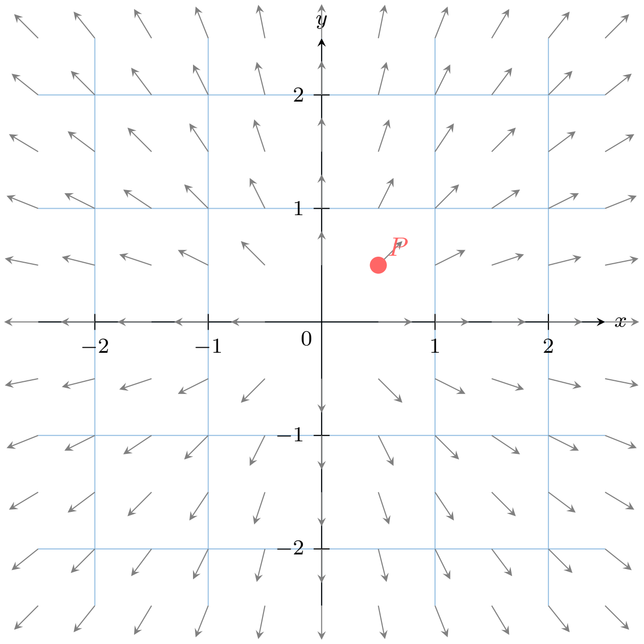
Ex 14: The phase portrait for a system of coupled differential equations is given below.
The vectors indicate the direction of motion at each point.
Starting from point $P(2, 0)$, sketch the **trajectory** of the system. Follow the flow of the arrows carefully.



Ex 15: The phase portrait for a system of coupled differential equations is given below.
The vectors indicate the direction of motion at each point.
Starting from point $P(-0.5, 2.5)$, sketch the **trajectory** of the system. Follow the flow of the arrows carefully.



Ex 16: The phase portrait for a system of coupled differential equations is given below.
The vectors indicate the direction of motion at each point.
Starting from point $P(0.5, 0.5)$, sketch the **trajectory** of the system. Follow the flow of the arrows carefully.



C COUPLED LINEAR DIFFERENTIAL EQUATIONS

C.1 WRITING COUPLED SYSTEMS IN MATRIX FORM

Ex 17: Consider the system of coupled linear differential equations:

$$\begin{cases} \frac{dx}{dt} = 2x - y \\ \frac{dy}{dt} = 3x + 2y \end{cases}$$

Let $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$.
Find the matrix $\mathbf{A}$ such that $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

Ex 18: Consider the system of coupled linear differential equations:

$$\begin{cases} \frac{dx}{dt} = 4x + 7y \\ \frac{dy}{dt} = -2x + 5y \end{cases}$$

Let $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$.
Find the matrix $\mathbf{A}$ such that $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

Ex 19: Consider the system of coupled linear differential equations:

$$\begin{cases} \frac{dx}{dt} = -x + 3y \\ \frac{dy}{dt} = 6x \end{cases}$$

Let $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$.
Find the matrix $\mathbf{A}$ such that $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

Ex 20: Consider the system of coupled linear differential equations:

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x - y \end{cases}$$

Let $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$.
Find the matrix $\mathbf{A}$ such that $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

C.2 SOLVING COUPLED LINEAR DIFFERENTIAL EQUATIONS

Ex 21: Consider the system of coupled differential equations:

$$\begin{cases} \frac{dx}{dt} = 4x - 2y \\ \frac{dy}{dt} = x + y \end{cases}$$

1. Write the system in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.
2. Find the eigenvalues λ_1 and λ_2 of the matrix $\mathbf{A}$.
3. Find the eigenvector $\mathbf{v}_1$ corresponding to λ_1 and the eigenvector $\mathbf{v}_2$ corresponding to λ_2.
4. Hence, write the general solution for the system.

Ex 22: Consider the system of coupled differential equations:

$$\begin{cases} \frac{dx}{dt} = 5x + 4y \\ \frac{dy}{dt} = x + 2y \end{cases}$$

1. Write the system in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.
2. Find the eigenvalues λ_1 and λ_2 of the matrix $\mathbf{A}$.
3. Find the eigenvector $\mathbf{v}_1$ corresponding to λ_1 and the eigenvector $\mathbf{v}_2$ corresponding to λ_2.

4. Hence, write the general solution for the system.

Ex 23: Consider the system of coupled differential equations:

$$\begin{cases} \dfrac{dx}{dt} = 4x + 2y \\ \dfrac{dy}{dt} = 3x - y \end{cases}$$

1. Write the system in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.
2. Find the eigenvalues λ_1 and λ_2 of the matrix $\mathbf{A}$.
3. Find the eigenvector $\mathbf{v}_1$ corresponding to λ_1 and the eigenvector $\mathbf{v}_2$ corresponding to λ_2.
4. Hence, write the general solution for the system.

D SECOND ORDER DIFFERENTIAL EQUATIONS

D.1 CONVERTING SECOND-ORDER EQUATIONS TO SYSTEMS

Ex 24: Consider the second-order differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$$

1. Convert into a system of first-order coupled equations.
2. Write the system in matrix form.

Ex 25: Consider the second-order differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$

1. Convert into a system of first-order coupled equations.
2. Write the system in matrix form.

Ex 26: Consider the second-order differential equation

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 3x = 0$$

1. Convert into a system of first-order coupled equations.
2. Write the system in matrix form.

Ex 27: When an object with mass m falls under gravity, its displacement x is governed by the equation of motion:

$$m\ddot{x} = -mg - k(\dot{x})^2$$

where $g \approx 9.8$ is the acceleration due to gravity and k is a positive constant representing air resistance.
Use the substitution $v = \dot{x}$ to write this as a coupled system of differential equations involving x and v.

D.2 MODELLING PHYSICAL SYSTEMS WITH SECOND-ORDER EQUATIONS

Ex 28: The temperature x of a drink t seconds after ice cubes are added to it, satisfies the differential equation

$$50\frac{d^2x}{dt^2} + \frac{dx}{dt} = 0$$

1. Use the substitution $y = \dfrac{dx}{dt}$ to write this as a coupled system of first order differential equations.
2. The equation for $\dfrac{dy}{dt}$ is separable and independent of x. Solve this equation for $y(t)$.
3. Hence find a general solution for $x(t)$.

Ex 29: The charge $q(t)$ (in coulombs) on the capacitor in a series RLC circuit satisfies the differential equation

$$20\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 100q = 0,$$

where t is time in seconds.

1. Use the substitution $i = \dfrac{dq}{dt}$ (where i represents the current in the circuit) to rewrite this equation as a coupled system of first-order differential equations.
2. The equation for $\dfrac{di}{dt}$ is separable and independent of q. Solve this equation to find $i(t)$.
3. Hence obtain the general solution for $q(t)$.

E EULER'S METHOD FOR COUPLED SYSTEMS

E.1 APPLYING EULER'S METHOD STEP BY STEP

Ex 30: Consider the system

$$\begin{cases} \dfrac{dx}{dt} = x - y \\ \dfrac{dy}{dt} = x + y \end{cases}$$

with initial conditions $x(0) = 1$, $y(0) = 0$. Use Euler's method with step size $h = 0.2$ to approximate $x(0.4)$ and $y(0.4)$ after two steps.

Ex 31: Consider the system

$$\begin{cases} \dfrac{dx}{dt} = -2x + y \\ \dfrac{dy}{dt} = x - 3y \end{cases}$$

with initial conditions $x(0) = 2$, $y(0) = 1$. Use Euler's method with step size $h = 0.1$ to find approximations for $x(0.2)$ and $y(0.2)$ after two steps.

Ex 32: Consider the non-linear system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x + y^2 \end{cases}$$

with initial conditions $x(0) = 0$, $y(0) = 1$. Use Euler's method with step size $h = 0.1$ to approximate $x(0.3)$ and $y(0.3)$ after three steps.