COUNTING

A BASIC COUNTING PRINCIPLES

A.1 APPLYING THE PRODUCT RULE

Ex 1: You have 3 t-shirts and 2 pairs of jeans. How many different outfits can you create by picking one t-shirt and one pair of jeans?

$\boxed{6}$ outfits

Answer: To find the total number of outfits, we multiply the number of options for each independent choice.

- Choice of a t-shirt: 3 options.
- Choice of jeans: 2 options.

By the product rule, the total number of outfits is $3 \times 2 = 6$.

Ex 2: You are at an ice cream shop with 5 flavors of ice cream and 2 types of toppings. How many different desserts can you create by picking one ice cream flavor and one topping?

$\boxed{10}$ desserts

Answer: The choices for the ice cream and topping are independent.

- Choice of ice cream flavor: 5 options.
- Choice of topping: 2 options.

By the product rule, the total number of different desserts is $5 \times 2 = 10$.

Ex 3: Su is creating a 4-digit PIN for his debit card. Each digit can be any number from 0 to 9, and digits can be repeated. His PIN starts with 94.

9	4		

How many different PINs are possible for the remaining two digits?

$\boxed{100}$ PINs

Answer: The first two digits are fixed. We must determine the number of possibilities for the third and fourth digits.

- Choices for the third digit (0-9): 10 options.
- Choices for the fourth digit (0-9): 10 options.

By the product rule, the number of possible PINs for the remaining digits is $10 \times 10 = 100$.

Ex 4: A headteacher wants to select one student from Year 4 and one student from Year 5 for an interview. There are 20 students in Year 4 and 18 students in Year 5. How many different pairs of students can be chosen?

$\boxed{360}$ pairs

Answer: The selection from each year is an independent event.

- Options for the Year 4 student: 20 choices.
- Options for the Year 5 student: 18 choices.

By the product rule, the total number of ways to choose the pair of students is $20 \times 18 = 360$.

Ex 5: You are packing for a trip with 2 pairs of shoes, 3 pairs of pants, and 5 t-shirts. How many different outfits can you create by picking one of each item?

$\boxed{30}$ outfits

Answer: This problem involves three independent choices. The product rule can be extended.

- Choice of shoes: 2 options.
- Choice of pants: 3 options.
- Choice of t-shirt: 5 options.

The total number of different outfits is $2 \times 3 \times 5 = 30$.

A.2 APPLYING THE ADDITION RULE

Ex 6: A student can choose a new computer from either 3 desktop models or 4 laptop models. What is the total number of computer options?

$\boxed{7}$ options

Answer: Let's solve this:

- The student must choose either a desktop OR a laptop. The two choices are mutually exclusive.
- Number of desktop options: 3.
- Number of laptop options: 4.
- Using the addition rule, we add the number of options for each exclusive choice:

$$3 \text{ (desktops)} + 4 \text{ (laptops)} = 7$$

- So, the total number of computer options is 7.

Ex 7: To travel from Paris to Lyon, a traveler can choose from 6 different high-speed trains or 2 flights. How many different travel options does the traveler have for their journey?

$\boxed{8}$ options

Answer:

- The traveler must choose either a train OR a flight; they cannot take both at the same time. The choices are mutually exclusive.
- Number of train options: 6.
- Number of flight options: 2.
- By the addition rule, the total number of options is the sum of the choices:

$$6 \text{ (trains)} + 2 \text{ (flights)} = 8$$

- Thus, the traveler has 8 different options for the journey.

Ex 8: A film enthusiast wants to watch a movie. They subscribe to two streaming services. Service A has 8 exclusive action movies, and Service B has 5 exclusive comedy movies. How many different movies can they choose to watch?

$$\boxed{13} \text{ movies}$$

Answer:

- The choice is between watching an action movie from Service A OR a comedy from Service B. Since the movies are exclusive to each service, the choices are mutually exclusive.
- Number of choices on Service A: 8.
- Number of choices on Service B: 5.
- Using the addition rule, we find the total number of available movies:

$$8 \text{ (action movies)} + 5 \text{ (comedies)} = 13$$

- Therefore, they can choose from 13 different movies.

Ex 9: A community center offers weekend workshops. On Saturday, there are 3 painting workshops and 2 pottery workshops. On Sunday, there are 4 creative writing workshops. If a person can only sign up for one workshop for the entire weekend, how many choices do they have?

$$\boxed{9} \text{ choices}$$

Answer:

- A person can choose a workshop on Saturday OR a workshop on Sunday. These are mutually exclusive time slots. First, let's find the total options for Saturday.
- Saturday options: 3 (painting) + 2 (pottery) = 5 workshops.
- Sunday options: 4 (writing) workshops.
- Now, we apply the addition rule again for the choice of day:

$$5 \text{ (Saturday workshops)} + 4 \text{ (Sunday workshops)} = 9$$

- So, the person has a total of 9 workshop choices for the weekend.

B FACTORIALS

B.1 EVALUATING WITHOUT A CALCULATOR

Ex 10: Evaluate:

$$3! = \boxed{6}.$$

Answer:

$$\begin{aligned} 3! &= 3 \times 2 \times 1 \\ &= 6 \end{aligned}$$

Ex 11: Evaluate:

$$4! = \boxed{24}$$

Answer:

$$\begin{aligned} 4! &= 4 \times 3 \times 2 \times 1 \\ &= 24 \end{aligned}$$

Ex 12: Evaluate:

$$5! = \boxed{120}$$

Answer:

$$\begin{aligned} 5! &= 5 \times 4 \times 3 \times 2 \times 1 \\ &= 120 \end{aligned}$$

Ex 13: Evaluate:

$$6! = \boxed{720}$$

Answer:

$$\begin{aligned} 6! &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 720 \end{aligned}$$

B.2 EVALUATING WITH A CALCULATOR

Ex 14: Evaluate:

$$7! = \boxed{5040}$$

Answer: Enter $7!$ in the calculator.

Ex 15: Evaluate:

$$\frac{8!}{3!} = \boxed{6720}$$

Answer: Enter $\frac{8!}{3!}$ in the calculator.

Ex 16: Evaluate:

$$\frac{9!}{3!6!} = \boxed{84}$$

Answer: Enter $\frac{9!}{3! \times 6!}$ in the calculator.

Ex 17: Evaluate:

$$\binom{20}{17} = \boxed{1140}$$

Answer:

- $\binom{20}{17} = \dfrac{20!}{17!(20-17)!} = \dfrac{20!}{17!3!}$
- Enter $\dfrac{20!}{17!3!}$ in the calculator.

Ex 18: Evaluate:

$$\binom{15}{10} = \boxed{3003}$$

Answer:

- $\binom{15}{10} = \dfrac{15!}{10!(15-10)!} = \dfrac{15!}{10!5!}$
- Enter $\dfrac{15!}{10!5!}$ in the calculator.

B.3 EXPRESSING PRODUCTS IN FACTORIAL FORM

Ex 19: Express in factorial form:

$$\frac{4\times 3\times 2\times 1}{2\times 1}=\boxed{\frac{4!}{2!}}$$

Answer:

$$\frac{4\times 3\times 2\times 1}{2\times 1}=\frac{4!}{2!}$$

Ex 20: Express in factorial form:

$$4\times 3=\boxed{\frac{4!}{2!}}$$

Answer:

$$\begin{aligned}4\times 3&=\frac{4\times 3\times (2\times 1)}{(2\times 1)}\\&=\frac{4!}{2!}\end{aligned}$$

Ex 21: Express in factorial form:

$$\frac{5\times 4\times 3\times 2\times 1}{3\times 2\times 1}=\boxed{\frac{5!}{3!}}$$

Answer:

$$\frac{5\times 4\times 3\times 2\times 1}{3\times 2\times 1}=\frac{5!}{3!}$$

Ex 22: Express in factorial form:

$$5\times 4=\boxed{\frac{5!}{3!}}$$

Answer:

$$\begin{aligned}5\times 4&=\frac{5\times 4\times (3\times 2\times 1)}{(3\times 2\times 1)}\\&=\frac{5!}{3!}\end{aligned}$$

Ex 23: Express in factorial form:

$$\frac{7\times 6\times 5\times 4\times 3\times 2\times 1}{4\times 3\times 2\times 1}=\boxed{\frac{7!}{4!}}$$

Answer:

$$\frac{7\times 6\times 5\times 4\times 3\times 2\times 1}{4\times 3\times 2\times 1}=\frac{7!}{4!}$$

Ex 24: Express in factorial form:

$$7\times 6\times 5=\boxed{\frac{7!}{4!}}$$

Answer:

$$\begin{aligned}7\times 6\times 5&=\frac{7\times 6\times 5\times (4\times 3\times 2\times 1)}{(4\times 3\times 2\times 1)}\\&=\frac{7!}{4!}\end{aligned}$$

B.4 EVALUATING BY SIMPLIFICATION

Ex 25: Evaluate

$$\binom{5}{3}=\boxed{10}$$

Answer:

$$\begin{aligned}\binom{5}{3}&=\frac{5!}{3!(5-3)!}\\&=\frac{5!}{3!2!}\\&=\frac{5\times 4\times \cancel{3\times 2\times 1}}{\cancel{3\times 2\times 1}\times 2\times 1}\\&=\frac{5\times 4}{2\times 1}\\&=10\end{aligned}$$

Ex 26: Evaluate

$$\binom{6}{4}=\boxed{15}$$

Answer:

$$\begin{aligned}\binom{6}{4}&=\frac{6!}{4!(6-4)!}\\&=\frac{6!}{4!2!}\\&=\frac{6\times 5\times \cancel{4\times 3\times 2\times 1}}{\cancel{4\times 3\times 2\times 1}\times 2\times 1}\\&=\frac{6\times 5}{2\times 1}\\&=15\end{aligned}$$

Ex 27: Evaluate

$$\binom{7}{2}=\boxed{21}$$

Answer:

$$\begin{aligned}\binom{7}{2}&=\frac{7!}{2!(7-2)!}\\&=\frac{7!}{2!5!}\\&=\frac{7\times 6\times \cancel{5\times 4\times 3\times 2\times 1}}{2\times 1\times \cancel{5\times 4\times 3\times 2\times 1}}\\&=\frac{7\times 6}{2\times 1}\\&=21\end{aligned}$$

Ex 28: Evaluate

$$\binom{7}{4}=\boxed{35}$$

Answer:

$$\begin{aligned}\binom{7}{4}&=\frac{7!}{4!(7-4)!}\\&=\frac{7!}{4!3!}\\&=\frac{7\times 6\times 5\times \cancel{4\times 3\times 2\times 1}}{\cancel{4\times 3\times 2\times 1}\times 3\times 2\times 1}\\&=\frac{7\times 6\times 5}{3\times 2\times 1}\\&=\frac{7\times \cancel{6}\times 5}{\cancel{6}}\\&=7\times 5\\&=35\end{aligned}$$

C ORDERED DRAWS WITH REPLACEMENT (P-LISTS)

C.1 SOLVING REAL-WORLD PROBLEMS

Ex 29: A suitcase lock requires a 3-digit code, with each digit ranging from 0 to 9. How many different codes are possible?

$\boxed{1000}$ codes

Answer: Let's break down the problem by considering each digit as a separate choice.

- For the first digit, there are 10 possible options (0 through 9).
- For the second digit, since digits can be repeated, there are also 10 possible options.
- For the third digit, there are again 10 possible options.

This scenario is equivalent to drawing 3 balls from a bag of 10 balls, where the **order matters** and each ball is put back before the next draw (**with replacement**).
Since the choice for each digit is independent, we use the product rule:

$$10 \text{ (choices for 1st digit)} \times 10 \text{ (choices for 2nd digit)} \times 10 \text{ (choices for 3rd digit)} = 10 \times 10 \times 10 = 1,000.$$

So, there are 1,000 different possible codes for the lock.

Ex 30: A mobile phone requires a 4-digit PIN, with each digit ranging from 0 to 9. How many different PINs can be created?

$\boxed{10000}$ PINs

Answer: We need to determine the number of options for each of the four positions in the PIN.

- There are 10 choices (0-9) for the first digit.
- There are 10 choices for the second digit.
- There are 10 choices for the third digit.
- There are 10 choices for the fourth digit.

This is a selection where order matters and repetition is allowed, equivalent to drawing 4 balls from a bag of 10, **with order and with replacement**.
Applying the product rule, we multiply the number of options for each position:

$$10 \times 10 \times 10 \times 10 = 10,000.$$

So, there are 10,000 different possible PINs.

Ex 31: You are taking a 5-question True/False test. How many different ways can you answer the entire test?

$\boxed{32}$ ways

Answer: For each question, there are two possible answers. We must make a choice for all five questions.

- For question 1, there are 2 choices (True or False).
- For question 2, there are 2 choices.
- For question 3, there are 2 choices.
- For question 4, there are 2 choices.
- For question 5, there are 2 choices.

This is analogous to drawing 5 times from a bag containing 2 balls (labeled 'T' and 'F'), **with order and with replacement**.
By the product rule, the total number of ways to answer the test is the product of the number of choices for each question:

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32.$$

So, there are 32 different ways to answer the test.

Ex 32: A multiple-choice test has 20 questions, and each question has 3 answer choices (A, B, or C). How many different answer keys are possible?

$\boxed{3486784401}$ answer keys

Answer: Each question is an independent choice, and the same answer option can be used for multiple questions.

- For each of the 20 questions, there are 3 possible answers (A, B, or C).

This corresponds to drawing 20 times from a bag of 3 balls (labeled 'A', 'B', 'C'), **with order and with replacement**.
Using the product rule, we multiply the number of choices for each of the 20 questions:

$$\underbrace{3 \times 3 \times \cdots \times 3}_{20 \text{ times}} = 3^{20} = 3,486,784,401.$$

So, there are 3,486,784,401 different possible answer keys.

D ORDERED DRAWS WITHOUT REPLACEMENT (ARRANGEMENTS)

D.1 SOLVING REAL-WORLD PROBLEMS

Ex 33: You're watching a race with 20 competitors, and the top three finishers will stand on the podium. How many different podiums (first, second, and third place) are possible?

$\boxed{6840}$ podiums

Answer: We need to select 3 competitors from 20 and assign them to a specific place.

- For first place, there are 20 possible competitors.
- Once first place is decided, there are 19 competitors remaining for second place.
- Once second place is decided, there are 18 competitors remaining for third place.

This scenario requires selecting competitors in a specific order, and a competitor cannot finish in more than one position. This is equivalent to drawing 3 balls from a bag of 20, where the **order matters** and there is **no replacement**. This is an **arrangement**.
Using the product rule, the total number of possible podiums is:

$$20 \text{ (choices for 1st)} \times 19 \text{ (choices for 2nd)} \times 18 \text{ (choices for 3rd)} = 6{,}840.$$

This corresponds to the formula for arrangements: ${}_{20}P_3 = 6,840$.

Ex 34: You're organizing a group photo and need to arrange 5 people in a single row. How many different ways can you line them up?

$\boxed{120}$ ways

Answer: We are arranging all 5 people in a specific order.

- For the first position in the row, there are 5 choices.
- For the second position, there are 4 remaining people.
- For the third position, there are 3 remaining people.
- For the fourth position, there are 2 remaining people.
- For the last position, there is only 1 person left.

This is an arrangement of all elements in a set, which is a **permutation**. It is equivalent to drawing all 5 balls from a bag of 5, **without replacement and with order**.
By the product rule, the total number of arrangements is:

$$5 \times 4 \times 3 \times 2 \times 1 = 5! = 120.$$

There are 120 different ways to arrange the 5 people.

Ex 35: You're watching a race with 20 competitors, including Emile. How many different possible podiums (first, second, and third place) are there if Emile must finish in first place?

$\boxed{342}$ podiums

Answer: This is an arrangement with a restriction.

- For first place, there is only 1 choice (Emile). The position is fixed.
- For second place, any of the other 19 competitors can finish. So, there are 19 choices.
- For third place, there are 18 competitors remaining. So, there are 18 choices.

After fixing Emile's position, the problem becomes arranging 2 competitors from the remaining 19. This is an **arrangement** of 2 from 19.
Using the product rule for the three positions:

$$1 \text{ (choice for 1st)} \times 19 \text{ (choices for 2nd)} \times 18 \text{ (choices for 3rd)} = 342.$$

There are 342 possible podiums where Emile finishes first.

Ex 36: You're watching a race with 20 competitors, including Emile. How many different podiums are possible if Emile must be one of the three podium finishers?

$\boxed{1026}$ podiums

Answer: We can solve this by first placing Emile and then arranging the others.

- First, choose a position for Emile. There are 3 choices (1st, 2nd, or 3rd).
- Once Emile's position is chosen, we must fill the two remaining podium spots from the other 19 competitors.
- For the first empty spot, there are 19 choices.
- For the second empty spot, there are 18 choices.

Using the product rule for the entire process:

$$3 \text{ (choices for Emile's position)} \times 19 \text{ (choices for 1st empty spot)} \times 18 \text{ (choices for 2nd empty spot)} = 1{,}026.$$

There are 1,026 possible podiums that include Emile.

MCQ 37: Mr. T has 5 algebra books, 3 geometry books, and 4 analysis books. In how many ways can he arrange them on a shelf if he groups them by subject?

- ☐ 12^{12}
- ☒ $3! \times 5! \times 3! \times 4!$
- ☐ $5! \times 3! \times 4!$
- ☐ $12!$

Answer: This problem has two levels of arrangement.

1. **Arrange the groups:** First, we arrange the 3 subject groups (Algebra, Geometry, Analysis). The number of ways to order these 3 groups is a permutation: $3! = 6$ ways.
2. **Arrange books within each group:**
   - The 5 algebra books can be arranged in 5! ways.
   - The 3 geometry books can be arranged in 3! ways.
   - The 4 analysis books can be arranged in 4! ways.

Using the product rule, we multiply the number of ways for each independent step:

$$\text{(Ways to arrange groups)} \times \text{(Ways to arrange algebra books)} \times \text{(Ways to arrange geometry books)} \times \text{(Ways to arrange analysis books)} = 3! \times 5! \times 3! \times 4!$$

The correct answer is $3! \times 5! \times 3! \times 4!$.

E UNORDERED DRAWS WITHOUT REPLACEMENT (COMBINATIONS)

E.1 SOLVING REAL-WORLD PROBLEMS

Ex 38: You are on a sports team with 5 players: P, Q, R, S, and T. How many different teams of 2 players can be formed?

$\boxed{10}$ teams

Answer: The problem is to choose a group of 2 players from 5. Since a team of {P, Q} is the same as a team of {Q, P}, the **order of selection does not matter**. Also, a player cannot be chosen twice for the same team, so this is **without replacement**. This is a **combination** problem.
This is analogous to drawing 2 balls from a bag of 5 balls (representing the players) **without order and without replacement**.

We use the combination formula to find the number of ways to choose 2 players from 5:

$$\begin{aligned}\binom{5}{2} &= \frac{5!}{2!(5-2)!}\\ &= \frac{5!}{2!3!}\\ &= \frac{5\times 4\times \cancel{3!}}{\cancel{3!}\times(2\times 1)}\\ &= \frac{20}{2} = 10\end{aligned}$$

There are 10 different teams of 2 players that can be formed.

Ex 39: You are part of a squad with 6 players. How many different teams of 4 players can be formed?

$\boxed{15}$ teams

Answer: We are choosing a group of 4 from 6 where the order does not matter and there is no repetition. This is a **combination**. It is analogous to drawing 4 balls from a bag of 6, **without order and without replacement**.
We use the combination formula to find the number of ways to choose 4 players from 6:

$$\begin{aligned}\binom{6}{4} &= \frac{6!}{4!(6-4)!}\\ &= \frac{6!}{4!2!}\\ &= \frac{6\times 5\times \cancel{4!}}{\cancel{4!}\times(2\times 1)}\\ &= \frac{30}{2} = 15\end{aligned}$$

There are 15 different teams of 4 players that can be formed.

Ex 40: How many different anagrams can be formed by rearranging the letters of the word "TOTO"?

$\boxed{6}$ anagrams

Answer: An anagram is determined by the positions of its letters. The word "TOTO" has 4 positions. We need to place two 'T's and two 'O's.Since the two 'T's are identical, the order in which we choose their positions does not matter. The problem is equivalent to choosing 2 positions for the 'T's out of the 4 available spots. This is a **combination**.
This is analogous to a bag with 4 balls numbered 1, 2, 3, 4 (representing the positions). We choose 2 balls **without order** to decide where the 'T's go.

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = \frac{24}{4} = 6$$

Once we choose the 2 positions for the 'T's, the remaining 2 positions are automatically filled by the 'O's. Therefore, there are 6 different anagrams.

Ex 41: In a class of 15 students, there are 9 boys and 6 girls. The teacher wants to form a group of 4 students, consisting of exactly 2 boys and 2 girls. How many different groups can be formed?

$\boxed{540}$ groups

Answer: This is a two-step problem. First, we choose the boys, and second, we choose the girls. In each step, the order of selection does not matter, so we use **combinations**.
The analogy is having two separate bags: one with 9 "boy" balls and one with 6 "girl" balls.

1. **Choose the boys:** We need to choose 2 boys from 9.

$$\text{Ways to choose boys} = \binom{9}{2} = \frac{9!}{2!7!} = \frac{9\times 8}{2} = 36$$

2. **Choose the girls:** We need to choose 2 girls from 6.

$$\text{Ways to choose girls} = \binom{6}{2} = \frac{6!}{2!4!} = \frac{6\times 5}{2} = 15$$

Since we need to choose both boys AND girls, we use the product rule to combine the results:

$$\begin{aligned}\text{Total groups} &= (\text{Ways to choose boys}) \times (\text{Ways to choose girls})\\ &= 36\times 15 = 540.\end{aligned}$$

There are 540 different ways to form the group.

Ex 42: Your company has 12 employees. A project team is to be formed with 1 team leader, 1 deputy leader, and 4 team members. How many different ways can this team be formed?

$\boxed{27720}$ ways

Answer: This problem involves both ordered and unordered selections. We can break it down into a sequence of choices.

1. **Choose the team leader:** The role of leader is unique. We are choosing 1 person from 12. There are 12 choices.
2. **Choose the deputy leader:** The role of deputy is also unique. From the remaining 11 employees, there are 11 choices.
3. **Choose the team members:** From the 10 remaining employees, we need to choose a group of 4. Since their roles are identical, **order does not matter**. This is a **combination**.

$$\text{Ways to choose members} = \binom{10}{4} = \frac{10!}{4!6!} = \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} = 2$$

Since these are sequential and independent choices, we apply the product rule:

$$(\text{Choices for Leader}) \times (\text{Choices for Deputy}) \times (\text{Choices for Members}) = 12\times 11\times 210 = 27,720.$$

There are 27,720 different ways to form the team.

F SUMMARY OF COUNTING FORMULAS

F.1 COUNTING POKER HANDS

Ex 43: A standard deck of 52 playing cards has 13 ranks and 4 suits. A poker hand consists of 5 cards drawn from this deck. How many different 5-card poker hands are possible?

$\boxed{2598960}$ hands

Answer: In a poker hand, the order in which the cards are received does not matter. Therefore, this is a **combination** problem: we are choosing a group of 5 cards from a set of 52.

$$\binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!} = 2,598,960$$

There are 2,598,960 different possible 5-card poker hands.

Ex 44: A **four-of-a-kind** is a hand with four cards of one rank and one card of another rank. How many different four-of-a-kind hands are possible?

$\boxed{624}$ hands

Answer: To construct this hand, we must make a sequence of choices.

1. **Choose the rank for the four cards:** From the 13 available ranks, we must choose 1.

$$\text{Ways to choose the rank} = \binom{13}{1} = 13$$

2. **Choose the four cards of that rank:** A rank has 4 suits. We must choose all 4 of them.

$$\text{Ways to choose the suits} = \binom{4}{4} = 1$$

3. **Choose the final card (the 'kicker'):** This card must not have the same rank as the four-of-a-kind. There are 52 cards in the deck, and 4 are already used, so 48 cards remain. We must choose 1 of them.

$$\text{Ways to choose the kicker} = \binom{48}{1} = 48$$

By the product rule, the total number of four-of-a-kind hands is:

$$13 \times 1 \times 48 = 624.$$

There are 624 possible four-of-a-kind hands.

Ex 45: A **full house** is a hand with three cards of one rank and two cards of another rank. How many different full house hands are possible?

$\boxed{3744}$ hands

Answer: We break down the construction into steps:

1. **Choose the rank for the three cards:** There are 13 ranks available.

$$\text{Ways to choose rank for the triplet} = \binom{13}{1} = 13$$

2. **Choose 3 cards from that rank:** There are 4 suits, and we need to choose 3.

$$\text{Ways to choose the 3 cards} = \binom{4}{3} = 4$$

3. **Choose the rank for the two cards:** There are 12 remaining ranks.

$$\text{Ways to choose rank for the pair} = \binom{12}{1} = 12$$

4. **Choose 2 cards from that rank:** There are 4 suits, and we need to choose 2.

$$\text{Ways to choose the 2 cards} = \binom{4}{2} = 6$$

Using the product rule, the total number of full house hands is:

$$13 \times 4 \times 12 \times 6 = 3,744.$$

There are 3,744 possible full house hands.

Ex 46: A **three-of-a-kind** is a hand with three cards of one rank, and two other cards of two different ranks. How many are possible?

$\boxed{54912}$ hands

Answer: Let's construct the hand step-by-step:

1. **Choose the rank for the three cards:**

$$\binom{13}{1} = 13$$

2. **Choose the 3 suits for that rank:**

$$\binom{4}{3} = 4$$

3. **Choose the ranks for the other two cards:** From the 12 remaining ranks, we must choose 2 different ranks. The order we choose them in doesn't matter.

$$\binom{12}{2} = \frac{12 \times 11}{2} = 66$$

4. **Choose the suit for each of the two single cards:** For the first single card, we choose 1 of 4 suits. For the second, we also choose 1 of 4 suits.

$$\binom{4}{1} \times \binom{4}{1} = 4 \times 4 = 16$$

By the product rule, the total number of three-of-a-kind hands is:

$$13 \times 4 \times 66 \times 16 = 54,912.$$

There are 54,912 possible three-of-a-kind hands.

Ex 47: A **two pair** hand has two cards of one rank, two cards of another rank, and one card of a third rank. How many are possible?

$\boxed{123552}$ hands

Answer: Let's build the hand:

1. **Choose the two ranks for the pairs:** From 13 ranks, we need to choose 2. The order does not matter (a pair of Kings and a pair of 7s is the same as a pair of 7s and a pair of Kings).
$$\binom{13}{2} = \frac{13 \times 12}{2} = 78$$
2. **Choose 2 suits for the first pair:**
$$\binom{4}{2} = 6$$
3. **Choose 2 suits for the second pair:**
$$\binom{4}{2} = 6$$
4. **Choose the rank for the final card:** There are $13 - 2 = 11$ ranks left.
$$\binom{11}{1} = 11$$
5. **Choose the suit for the final card:**
$$\binom{4}{1} = 4$$

Using the product rule, the total number of two-pair hands is:

$$78 \times 6 \times 6 \times 11 \times 4 = 123,552.$$

There are 123,552 possible two-pair hands.

Ex 48: A **one pair** hand has two cards of one rank, and three other cards of three different ranks. How many are possible?

$\boxed{1098240}$ hands

Answer: Let's construct the hand:

1. **Choose the rank for the pair:**
$$\binom{13}{1} = 13$$
2. **Choose 2 suits for that rank:**
$$\binom{4}{2} = 6$$
3. **Choose the ranks for the three single cards:** From the 12 remaining ranks, we must choose 3 different ranks.
$$\binom{12}{3} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220$$
4. **Choose the suit for each of the three single cards:**
$$\binom{4}{1} \times \binom{4}{1} \times \binom{4}{1} = 4^3 = 64$$

By the product rule, the total number of one-pair hands is:

$$13 \times 6 \times 220 \times 64 = 1,098,240.$$

There are 1,098,240 possible one-pair hands.

G APPLICATIONS TO PROBABILITY

G.1 CALCULATING PROBABILITIES USING COUNTING PRINCIPLES

Ex 49: A student council has 10 members: 6 seniors and 4 juniors. A 3-person committee is to be selected at random. What is the probability that the committee will consist entirely of seniors?

Answer:

1. **Identify the experiment:** We are selecting a group of 3 students from a larger group of 10.
2. **Model the outcomes:** When forming a committee, the order of selection does not matter. Therefore, the model is a **combination**.
3. **Calculate the total number of outcomes ($n(U)$):** We need to find the total number of ways to choose 3 students from the 10 members.
$$n(U) = \binom{10}{3} = \frac{10!}{3!(7)!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$$
There are 120 possible committees.
4. **Calculate the number of favorable outcomes ($n(E)$):** A favorable outcome is a committee that consists only of seniors. We need to choose 3 students from the 6 available seniors.
$$n(E) = \binom{6}{3} = \frac{6!}{3!(3)!} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$$
There are 20 possible committees made up entirely of seniors.
5. **Calculate the probability:**
$$P(\text{All Seniors}) = \frac{n(E)}{n(U)} = \frac{20}{120} = \frac{1}{6}$$

The probability of the committee consisting entirely of seniors is $1/6$.

Ex 50: In a race with 20 horses, you bet on 3 horses to finish first, second, and third in exact order (a "triple forecast"). What's the probability of winning your bet?

Answer:

1. **Identify the experiment:** We are determining the ordered result of the top three finishers in a 20-horse race.
2. **Model the outcomes:** We are selecting 3 horses from 20. Since the finishing order is exact, **order matters**. A horse cannot finish in more than one position, so this is **without replacement**. The model is an **arrangement**.
3. **Calculate the total number of outcomes ($n(U)$):** The total number of possible podium finishes is the number of arrangements of 3 horses from a set of 20.
$$n(U) = {}_{20}P_3 = \frac{20!}{(20-3)!} = 20 \times 19 \times 18 = 6,840$$
There are 6,840 possible ordered outcomes.

4. **Calculate the number of favorable outcomes ($n(E)$):** To win, your single bet must match the one correct finishing order.
$$n(E) = 1$$

5. **Calculate the probability:**
$$P(\text{Win}) = \frac{n(E)}{n(U)} = \frac{1}{6,840}$$

The probability of winning is 1 in 6,840.

Ex 51: In a race with 20 horses, you bet on 3 horses to finish in the top 3 positions in any order (a "trio" bet). What's the probability of winning your bet?

Answer:

1. **Identify the experiment:** We are determining the group of horses that finish in the top three, regardless of their specific order.

2. **Model the outcomes:** We are selecting a group of 3 horses from 20. Since the finishing order does not matter, **order does not matter**. This is **without replacement**. The model is a **combination**.

3. **Calculate the total number of outcomes ($n(U)$):** The total number of possible groups of three horses is the number of combinations of 3 from a set of 20.
$$n(U) = \binom{20}{3} = \frac{20!}{3!(17)!} = \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 1,140$$
There are 1,140 possible groups of top-three finishers.

4. **Calculate the number of favorable outcomes ($n(E)$):** Your bet is on one specific group of 3 horses. There is only one winning group.
$$n(E) = 1$$

5. **Calculate the probability:**
$$P(\text{Win}) = \frac{n(E)}{n(U)} = \frac{1}{1,140}$$

The probability of winning is 1 in 1,140.

Ex 52: In Lotto, you pick 6 numbers from a grid of 49. What is the probability of winning the jackpot by matching all 6 numbers?

Answer:

1. **Identify the experiment:** A set of 6 winning numbers is drawn from 49.

2. **Model the outcomes:** We are choosing a group of 6 numbers from 49. The order in which the numbers are drawn does not matter. This is a **combination**.

3. **Calculate the total number of outcomes ($n(U)$):** The total number of possible ways to draw 6 numbers from 49 is:
$$n(U) = \binom{49}{6} = \frac{49!}{6!(43)!} = 13,983,816$$
There are 13,983,816 possible lottery tickets.

4. **Calculate the number of favorable outcomes ($n(E)$):** There is only one set of 6 numbers that wins the jackpot.
$$n(E) = 1$$

5. **Calculate the probability:**
$$P(\text{Win}) = \frac{n(E)}{n(U)} = \frac{1}{13,983,816}$$

The probability of winning the jackpot is 1 in 13,983,816.

Ex 53: In Lotto, you pick 6 numbers from a grid of 49. What is the probability of matching exactly 5 of the 6 winning numbers?

Answer:

1. **Identify the experiment:** A set of 6 winning numbers is drawn from 49. We want to find the probability that our ticket of 6 numbers contains exactly 5 of these winning numbers.

2. **Model the outcomes:** Selections of numbers are **combinations**. We will need the product rule to construct the favorable outcomes.

3. **Calculate the total number of outcomes ($n(U)$):** As before, the total number of possible tickets is:
$$n(U) = \binom{49}{6} = 13,983,816$$

4. **Calculate the number of favorable outcomes ($n(E)$):** For a ticket to be a winner, it must contain 2 parts: 5 winning numbers AND 1 losing number.
   - There are 6 winning numbers in total, and we must choose 5 of them: $\binom{6}{5} = 6$ ways.
   - There are $49 - 6 = 43$ losing numbers, and we must choose 1 of them: $\binom{43}{1} = 43$ ways.

   By the product rule, the total number of favorable tickets is:
$$n(E) = \binom{6}{5} \times \binom{43}{1} = 6 \times 43 = 258$$

5. **Calculate the probability:**
$$P(\text{match } 5) = \frac{n(E)}{n(U)} = \frac{258}{13,983,816}$$

The probability of matching exactly 5 numbers is 258 in 13,983,816.

Ex 54: In a class of 30 students, what is the probability that at least two students share the same birthday? Assume a year has 365 days and birthdays are equally likely.

Answer:

1. **Identify the experiment:** We are assigning a birthday from 365 days to each of the 30 students.

2. **Model the outcomes:** For the total outcomes, each student's birthday is an independent choice from 365 days. This is a **p-list** (ordered, with replacement). For the favorable outcomes, we will use the complementary event.

3. **Calculate the total number of outcomes ($n(U)$):** Each of the 30 students has 365 possible birthdays.

$$n(U) = \underbrace{365 \times 365 \times \cdots \times 365}_{\text{30 times}} = 365^{30}$$

4. **Calculate the number of favorable outcomes ($n(E)$):** Let E be the event "at least two students share a birthday". It is easier to calculate the probability of the complementary event, E', where "all 30 students have different birthdays". To find the number of outcomes in E', we use an **arrangement**, since order matters (the students are distinct) and birthdays cannot be repeated.

$$n(E') = {}_{365}P_{30} = 365 \times 364 \times \cdots \times (365 - 29)$$

5. **Calculate the probability:** We first find the probability of the complement, P(E').

$$P(E') = \frac{n(E')}{n(U)} = \frac{{}_{365}P_{30}}{365^{30}} \approx 0.2937$$

The probability of our original event E is $1 - P(E')$.

$$P(E) = 1 - P(E') \approx 1 - 0.2937 = 0.7063$$

Rounded to two decimal places, the probability that at least two students share a birthday is 0.71.