COUNTING

A BASIC COUNTING PRINCIPLES

A.1 APPLYING PRODUCT RULE

Ex 1: Imagine you're getting ready for school and opening your cupboard. You have 3 t-shirts and 2 pairs of jeans to choose from. How many different outfits can you create by picking one t-shirt and one pair of jeans?

$\boxed{6}$ outfits

Answer: Let's break it down:

- You have 3 t-shirts and 2 pairs of jeans, so you have 3 choices for the t-shirt and 2 choices for the jeans.
- Using the product rule, which tells us to multiply the number of choices for each independent decision, we calculate:

$$3 \text{ (options for the t-shirt)} \times 2 \text{ (options for the jeans)} = 3 \times 2 = 6$$

- So, there are 6 different possible outfits you can create.

Ex 2: You're getting dressed for a day out and check your closet. You find 4 hats and 3 pairs of shoes to choose from. How many different outfits can you create by picking one hat and one pair of shoes?

$\boxed{12}$ outfits

Answer: Let's figure it out:

- You have 4 hats and 3 pairs of shoes, so you have 4 choices for the hat and 3 choices for the shoes.
- Using the product rule, we multiply the number of choices for each independent decision:

$$4 \text{ (options for the hat)} \times 3 \text{ (options for the shoes)} = 4 \times 3 = 12$$

- So, there are 12 different possible outfits you can create.

Ex 3: You're at an ice cream shop with 5 flavors of ice cream and 2 types of toppings. How many different dessert combinations can you create by picking one ice cream flavor and one topping?

$\boxed{10}$ combinations

Answer: Let's calculate it:

- You have 5 ice cream flavors and 2 topping options, so you have 5 choices for the ice cream and 2 choices for the topping.
- Using the product rule, we multiply the number of choices for each independent decision:

$$5 \text{ (options for the ice cream)} \times 2 \text{ (options for the topping)} = 5 \times 2 = 10$$

- So, there are 10 different possible dessert combinations you can create.

Ex 4: Su is creating a 4-digit PIN for his debit card. Each digit can be any number from 0 to 9, and he can repeat digits. His PIN starts with 94.

9	4		

How many different PIN codes are possible for the remaining two digits?

$\boxed{100}$ codes

Answer: Let's solve this:

- Since each digit can be any number from 0 to 9 and digits can be repeated, there are 10 possible numbers for each digit.
- The PIN already starts with 94, so we need to find the number of choices for the third digit and the fourth digit.
- We have 10 choices for the third digit and 10 choices for the fourth digit.
- Using the product rule, we multiply the number of choices for each independent digit:

$$10 \text{ (options for the third digit)} \times 10 \text{ (options for the fourth digit)} = 10 \times 10 = 100$$

- So, there are 100 different possible PIN codes for the remaining two digits.

Ex 5: At a school, there are 20 students in Year 4 and 18 students in Year 5. The headteacher wants to select one student from Year 4 and one student from Year 5 for an interview. How many different ways can the headteacher choose these two students?

$\boxed{360}$ ways

Answer: Let's work through it:

- There are 20 students in Year 4 and 18 students in Year 5, so the headteacher has 20 choices for the Year 4 student and 18 choices for the Year 5 student.
- Since choosing one student doesn't affect the choice of the other, these are independent decisions.
- Using the product rule, we multiply the number of choices for each grade:

$$20 \text{ (options for Year 4 student)} \times 18 \text{ (options for Year 5 student)} = 20 \times 18 = 360$$

- So, there are 360 different ways the headteacher can choose one student from Year 4 and one from Year 5.

Ex 6: You're packing for a trip and have 2 pairs of shoes, 3 pairs of pants, and 5 t-shirts to choose from. How many different outfits can you create by picking one pair of shoes, one pair of pants, and one t-shirt?

$\boxed{30}$ outfits

Answer: Let's figure it out:

- You have 2 pairs of shoes, 3 pairs of pants, and 5 t-shirts, so you have 2 choices for shoes, 3 choices for pants, and 5 choices for t-shirts.
- Since each choice is independent, we use the product rule to multiply the number of options for each item:

$$2 \text{ (options for shoes)} \times 3 \text{ (options for pants)} \times 5 \text{ (options for t-shirts)} = 2 \times 3 \times 5 = 30$$

- So, there are 30 different outfits you can create.

A.2 ADDITION RULE

Ex 7: A student is shopping for a new computer and can choose between 3 desktop models and 4 laptop models. What is the total number of computer options they can consider?

$\boxed{7}$ computer options

Answer: Let's solve this:

- The student can choose from 3 desktop computers or 4 laptop computers, but they will pick only one type (desktop or laptop).
- Using the addition rule, which adds the number of options when the choices are mutually exclusive, we calculate:

$$3 \text{ (options for desktop computers)} + 4 \text{ (options for laptop computers)} = 3 + 4 = 7$$

- So, the total number of computer options is 7.

Ex 8: A person is picking a movie to watch and can choose from 6 action movies or 4 comedy movies. What is the total number of movie options they can consider?

$\boxed{10}$ movie options

Answer: Let's work through it:

- The person can choose from 6 action movies or 4 comedy movies, but they will watch only one movie (either an action or a comedy).
- Using the addition rule, which adds the number of options when the choices are mutually exclusive, we calculate:

$$6 \text{ (options for action movies)} + 4 \text{ (options for comedy movies)} = 6 + 4 = 10$$

- So, the total number of movie options is 10.

Ex 9: At a café, a customer can choose between 4 types of coffee or 5 types of tea. What is the total number of beverage options they can consider?

$\boxed{9}$ beverage options

Answer: Let's figure it out:

- The customer can choose from 4 coffee options or 5 tea options, but they will order only one beverage (either coffee or tea).
- Using the addition rule, which adds the number of options when the choices are mutually exclusive, we calculate:

$$4 \text{ (options for coffee)} + 5 \text{ (options for tea)} = 4 + 5 = 9$$

- So, the total number of beverage options is 9.

Ex 10: A customer is shopping for a new phone and can choose between 5 smartphone models or 3 flip phone models. What is the total number of phone options they can consider?

$\boxed{8}$ phone options

Answer: Let's solve this:

- The customer can choose from 5 smartphone models or 3 flip phone models, but they will pick only one type (smartphone or flip phone).
- Using the addition rule, which adds the number of options when the choices are mutually exclusive, we calculate:

$$5 \text{ (options for smartphone models)} + 3 \text{ (options for flip phone models)} = 5 + 3 = 8$$

- So, the total number of phone options is 8.

B FACTORIALS

B.1 EVALUATING WITHOUT USING A CALCULATOR

Ex 11: Evaluate without using a calculator

$$3! = \boxed{6}.$$

Answer:

$$\begin{aligned} 3! &= 3 \times 2 \times 1 \\ &= 6 \end{aligned}$$

Ex 12: Evaluate without using a calculator

$$4! = \boxed{24}$$

Answer:

$$\begin{aligned} 4! &= 4 \times 3 \times 2 \times 1 \\ &= 24 \end{aligned}$$

Ex 13: Evaluate without using a calculator

$$5! = \boxed{120}$$

Answer:

$$\begin{aligned} 5! &= 5 \times 4 \times 3 \times 2 \times 1 \\ &= 120 \end{aligned}$$

Ex 14: Evaluate without using a calculator

$$6! = \boxed{720}$$

Answer:

$$\begin{aligned} 6! &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 720 \end{aligned}$$

B.2 EVALUATING WITH A CALCULATOR

Ex 15: Use a calculator to evaluate:

$$7! = \boxed{5040}$$

Answer: Enter 7! in the calculator.

Ex 16: Use a calculator to evaluate:

$$\frac{8!}{3!} = \boxed{6720}$$

Answer: Enter 8!/3! in the calculator.

Ex 17: Use a calculator to evaluate:

$$\frac{9!}{3!6!} = \boxed{84}$$

Answer: Enter 9!/(3!*6!) in the calculator.

Ex 18: Use a calculator to evaluate:

$$\binom{20}{17} = \boxed{1140}$$

Answer:

- $\binom{20}{17} = \frac{20!}{17!(20-3)!} = \frac{20!}{17!3!}$
- Enter 20!/(17!*3!) in the calculator.

Ex 19: Use a calculator to evaluate:

$$\binom{15}{10} = \boxed{3003}$$

Answer:

- $\binom{15}{10} = \frac{15!}{10!(15-10)!} = \frac{15!}{10!5!}$
- Enter 15!/(10!*5!) in the calculator.

B.3 EXPRESSING IN FACTORIAL FORM

Ex 20: Express in factorial form:

$$\frac{4\times3\times2\times1}{2\times1} = \boxed{\frac{4!}{2!}}$$

Answer:

$$\frac{4\times3\times2\times1}{2\times1} = \frac{4!}{2!}$$

Ex 21: Express in factorial form:

$$4\times3 = \boxed{\frac{4!}{2!}}$$

Answer:

$$\begin{aligned} 4\times3 &= \frac{4\times3\times2\times1}{2\times1} \\ \& &= \frac{4!}{2!} \end{aligned}$$

Ex 22: Express in factorial form:

$$\frac{5\times4\times3\times2\times1}{3\times2\times1} = \boxed{\frac{5!}{3!}}$$

Answer:

$$\frac{5\times4\times3\times2\times1}{3\times2\times1} = \frac{5!}{3!}$$

Ex 23: Express in factorial form:

$$5\times4 = \boxed{\frac{5!}{3!}}$$

Answer:

$$\begin{aligned} 5\times4 &= \frac{5\times4\times3\times2\times1}{3\times2\times1} \\ \& &= \frac{5!}{3!} \end{aligned}$$

Ex 24: Express in factorial form:

$$\frac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1} = \boxed{\frac{7!}{4!}}$$

Answer:

$$\frac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1} = \frac{7!}{4!}$$

Ex 25: Express in factorial form:

$$7\times6\times5 = \boxed{\frac{7!}{4!}}$$

Answer:

$$\begin{aligned} 7\times6\times5 &= \frac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1} \\ \& &= \frac{7!}{4!} \end{aligned}$$

B.4 BINOMIAL COEFFICIENT

Ex 26: Evaluate without using a calculator

$$\binom{5}{3} = \boxed{10}$$

Answer:

$$\begin{aligned} \binom{5}{3} &= \frac{5!}{3!(5-3)!} \\ &= \frac{5!}{3!2!} \\ &= \frac{5\times4\times\cancel{3\times2\times1}}{\cancel{3\times2\times1}\times2\times1} \\ &= \frac{5\times4}{2\times1} \\ &= 10 \end{aligned}$$

Ex 27: Evaluate without using a calculator

$$\binom{6}{4} = \boxed{15}$$

Answer:

$$\begin{aligned} \binom{6}{4} &= \frac{6!}{4!(6-4)!} \\ &= \frac{6!}{4!2!} \\ &= \frac{6\times5\times\cancel{4\times3\times2\times1}}{\cancel{4\times3\times2\times1}\times2\times1} \\ &= \frac{6\times5}{2\times1} \\ &= 15 \end{aligned}$$

Ex 28: Evaluate without using a calculator

$$\binom{7}{2} = \boxed{21}$$

Answer:

$$\begin{aligned}\binom{7}{2} &= \frac{7!}{2!(7-2)!}\\ &= \frac{7!}{2!5!}\\ &= \frac{7\times 6\times \cancel{5\times 4\times 3\times 2\times 1}}{2\times 1\times \cancel{5\times 4\times 3\times 2\times 1}}\\ &= \frac{7\times 6}{2\times 1}\\ &= 21\end{aligned}$$

Ex 29: Evaluate without using a calculator

$$\binom{7}{4} = \boxed{35}$$

Answer:

$$\begin{aligned}\binom{7}{4} &= \frac{7!}{4!(7-4)!}\\ &= \frac{7!}{4!3!}\\ &= \frac{7\times 6\times 5\times \cancel{4\times 3\times 2\times 1}}{\cancel{4\times 3\times 2\times 1}\times 3\times 2\times 1}\\ &= \frac{7\times 6\times 5}{3\times 2\times 1}\\ &= \frac{7\times \cancel{6}\times 5}{\cancel{6}}\\ &= 7\times 5\\ &= 35\end{aligned}$$

C ORDERED DRAWS WITH REPLACEMENT

C.1 SOLVING REAL-WORLD PROBLEMS

Ex 30: You're packing for a trip and need to set a combination lock on your suitcase. The lock uses a 3-digit code, with each digit ranging from 0 to 9. How many different combinations can you create for the lock?

$\boxed{1000}$ combinations

Answer: Let's break it down:

- Each digit in the 3-digit code can be any number from 0 to 9, so there are 10 possible choices for each digit. With three positions, this is like drawing three balls in order, with replacement, from a bag of 10 numbered balls.
- Using the product rule, we multiply the number of choices for each digit:

  10 (options for the first digit) $\times$ 10 (options for the second digit) $\times$ 10 (options for the third digit) $= 10^3 = 1,000$

- So, there are 1,000 different possible combinations for the suitcase lock.

Ex 31: You're setting up a security code for your mobile phone, which requires a 4-digit PIN, with each digit ranging from 0 to 9. How many different codes can you create?

$\boxed{10000}$ codes

Answer: Let's figure it out:

- Each digit in the 4-digit PIN can be any number from 0 to 9, so there are 10 possible choices for each digit. With four positions, this is like drawing four balls in order, with replacement, from a bag of 10 numbered balls.
- Using the product rule, we multiply the number of choices for each digit:

  10 (options for the first digit) $\times$ 10 (options for the second digit) $\times$ 10 (options for the third digit) $\times$ 10 (options for the fourth digit) $= 10^4 = 10,000$

- So, there are 10,000 different possible codes for the mobile phone.

Ex 32: You're taking a 5-question True/False test in class. Each question has two possible answers: True or False. How many different ways can you answer the test?

$\boxed{32}$ ways

Answer: Let's work through it:

- Each of the 5 questions has 2 possible answers (True or False), so there are 2 choices for each question. This is like drawing five balls in order, with replacement, from a bag of 2 numbered balls.
- Using the product rule, we multiply the number of choices for each question:

  2 (options for the first question) $\times$ 2 (options for the second question) $\times$ 2 (options for the third question) $\times$ 2 (options for the fourth question) $\times$ 2 (options for the fifth question) $= 2^5 = 32$

- So, there are 32 different possible ways to answer the 5-question True/False test.

Ex 33: You're taking a multiple-choice test with 20 questions, and each question has 3 answer choices (A, B, or C). How many distinctly different answer keys are possible? (Use a calculator to compute this.)

$\boxed{3486784401}$ answer keys

Answer: Let's solve this:

- Each of the 20 questions has 3 possible answers (A, B, or C), so there are 3 choices for each question. This is like drawing twenty balls in order, with replacement, from a bag of 3 numbered balls.
- Using the product rule, we multiply the number of choices for each question. Since the number is large, we use a calculator:

  3 (options for the first question) $\times$ 3 (options for the second question) $\times \cdots \times$ 3 (options for the twentieth question) $= 3^{20} = 3,486,784,401$

- So, there are 3,486,784,401 different possible answer keys for the 20-question multiple-choice test.

D ORDERED DRAWS WITHOUT REPLACEMENT

D.1 SOLVING REAL-WORLD PROBLEMS

Ex 34: You're watching a race with 20 competitors, and the top three finishers will stand on the podium. How many different possible podiums (first, second, and third place) can there be? (Use a calculator to compute this.)

$\boxed{6840}$ podiums

Answer: Let's break it down:

- A podium corresponds to selecting the top three finishers in order (first, second, and third place) from the 20 competitors, without any competitor finishing in multiple positions. This is like drawing 3 balls in order, without replacement, from a bag of 20 numbered balls.
- For first place, there are 20 possible choices. For second place, after one competitor is chosen, there are 19 remaining choices. For third place, there are 18 remaining choices.
- Using the product rule, the total number of possible podiums is:

$$20 \text{ (options for first place)} \times 19 \text{ (options for second place)} \times 18 \text{ (options for third place)} = 20 \times 19 \times 18 = 6,840$$

- So, there are 6,840 different possible podiums for the race.

Ex 35: You're organizing a group photo and need to arrange 5 people in a single row. How many different ways can you line them up? (Use a calculator to compute this.)

$\boxed{120}$ ways

Answer: Let's figure it out:

- Arranging 5 people in a row means assigning each person a unique position, with no repeats. This is like drawing 5 balls in order, without replacement, from a bag of 5 numbered balls.
- For the first position, there are 5 possible choices. For the second position, there are 4 remaining choices, for the third position, 3 choices, for the fourth position, 2 choices, and for the fifth position, 1 choice.
- Using the product rule, the total number of ways to arrange 5 people in a row is:

$$5 \text{ (options for the first position)} \times 4 \text{ (options for the second position)} \times 3 \text{ (options for the third position)} \times 2 \text{ (options for the fourth position)} \times 1 \text{ (options for the fifth position)} = 5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$$

- So, there are 120 different ways to arrange the 5 people in a row for the photograph.

Ex 36: King Art is hosting a feast at his round table, sitting at the head position (of course). His seven knights need to sit around him in a circle. How many different ways can the knights be arranged? (Use a calculator to compute this.)

$\boxed{5040}$ ways

Answer: Let's solve this:

- Arranging 7 knights in a circle around King Art (who is fixed at the head position) means assigning each knight a unique position in the circle, with no repeats. Since the table is round and King Art is fixed, we treat the knights as being arranged in a linear order relative to him. This is like drawing 7 balls in order, without replacement, from a bag of 7 numbered balls.
- For the first knight's position, there are 7 possible choices. For the second knight's position, there are 6 remaining choices, and so on, down to 1 choice for the last knight.
- Using the product rule, the total number of ways to arrange the 7 knights in a circle around King Art is:

$$7 \text{ (options for the first knight)} \times 6 \text{ (options for the second knight)} \times 5 \text{ (options for the third knight)} \times 4 \text{ (options for the fourth knight)} \times 3 \text{ (options for the fifth knight)} \times 2 \text{ (options for the sixth knight)} \times 1 \text{ (options for the seventh knight)} = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7! = 5,040$$

- So, there are 5,040 different ways to arrange the seven knights around King Art at the round table.

Ex 37: You're watching a race with 20 competitors, including Emile. How many different possible podiums (first, second, and third place) can there be if Emile must finish in first place? (Use a calculator to compute this.)

$\boxed{342}$ podiums

Answer: Let's work through it:

- Since Emile must finish in first place, we fix him as the first-place finisher. The remaining two podium positions (second and third place) need to be filled by the other 19 competitors, in order, without any repeats. This is like drawing 2 balls in order, without replacement, from a bag of 19 numbered balls.
- For second place, there are 19 possible choices from the remaining competitors. For third place, there are 18 remaining choices.
- Using the product rule, the total number of possible podiums with Emile in first place is:

$$19 \text{ (options for second place)} \times 18 \text{ (options for third place)} = 19 \times 18 = 342$$

- So, there are 342 different possible podiums where Emile finishes first in the race.

Ex 38: You're watching a race with 20 competitors, including Emile. How many different possible podiums (first, second, and third place) can there be if Emile must be one of the three podium finishers? (Use a calculator to compute this.)

$\boxed{1026}$ podiums

Answer: Let's solve this:

- Emile must be one of the three podium positions (first, second, or third place). There are 3 possible positions where Emile can finish on the podium.

- Once Emile's position is fixed, the remaining two podium positions need to be filled by the other 19 competitors, in order, without any repeats. This is like drawing 2 balls in order, without replacement, from a bag of 19 numbered balls.

- For each of the 3 positions Emile can take, there are 19 choices for the first remaining position and 18 choices for the second remaining position.

- Using the product rule, the total number of possible podiums with Emile on the podium is:

$$3 \text{ (options for Emile's position)} \times 19 \text{ (options for the first remaining position)} \times 18 \text{ (options for the second remaining position)} = 3 \times 19 \times 18 = 1,026$$

- So, there are 1,026 different possible podiums where Emile is one of the top three finishers in the race.

MCQ 39: Mr. T has 5 algebra books, 3 geometry books, and 4 analysis books. In how many ways can he arrange them on a shelf in his library?

☐ 12^{12}

☐ $3! \times 5! \times 3! \times 4!$

☐ $5! \times 3! \times 4!$

☒ $12!$

Answer: Let's solve this:

- Mr. T has 5 algebra books, 3 geometry books, and 4 analysis books, making a total of $5 + 3 + 4 = 12$ distinct books. Arranging them on a shelf means assigning each book a unique position, with no repeats. This is like drawing 12 balls in order, without replacement, from a bag of 12 numbered balls.

- The number of ways to arrange 12 distinct books is given by the factorial of 12, which is $12 \times 11 \times \cdots \times 2 \times 1 = 12!$.

- Therefore, the correct answer is 12!.

MCQ 40: Books Mr. T has 5 algebra books, 3 geometry books, and 4 analysis books. In how many ways can he arrange them on a shelf in his library by grouping them by subject?

☐ 12^{12}

☒ $3! \times 5! \times 3! \times 4!$

☐ $5! \times 3! \times 4!$

☐ $12!$

Answer: Let's figure it out:

- If Mr. T wants to arrange the books on the shelf by grouping them by subject (algebra, geometry, and analysis), he first needs to arrange the 3 groups in order. There are 3! ways to arrange these groups.

- Within each group, he needs to arrange the books: there are 5! ways to arrange the 5 algebra books, 3! ways to arrange the 3 geometry books, and 4! ways to arrange the 4 analysis books.

- Using the product rule, the total number of ways to arrange the books on the shelf, grouped by subject, is the product of the ways to arrange the groups and the ways to arrange the books within each group:

$$3! \times 5! \times 3! \times 4!$$

- Therefore, the correct answer is $3! \times 5! \times 3! \times 4!$.

MCQ 41: In a town, there are four bakeries that close one day a week. Determine the number of ways to assign a weekly closing day to each bakery.

☐ $4!$

☒ 7^4

☐ $7 \times 6 \times 5 \times 4$

Answer: Let's solve this:

- There are 4 bakeries, and each can close on any of the 7 days of the week. Since the closing days are independent, this is like drawing 4 balls in order, with replacement, from a bag of 7 numbered balls.

- Using the product rule, we multiply the number of choices for each bakery's closing day:

$$7 \text{ (choices for Bakery 1)} \times 7 \text{ (choices for Bakery 2)} \times 7 \text{ (choices for Bakery 3)} \times 7 \text{ (choices for Bakery 4)} = 7^4$$

- Therefore, the correct answer is 7^4.

MCQ 42: In a town, there are four bakeries that close one day a week. Determine the number of ways to assign a weekly closing day to each bakery if no two bakeries can close on the same day.

☐ $4!$

☐ 7^4

☒ $7 \times 6 \times 5 \times 4$

Answer: Let's figure it out:

- There are 4 bakeries, and each must close on a different day of the week (7 days available). This is like drawing 4 balls in order, without replacement, from a bag of 7 numbered balls.

- The first bakery can choose any of the 7 days, the second bakery can choose from the remaining 6 days, the third bakery from the remaining 5 days, and the fourth bakery from the remaining 4 days.

- Using the product rule, the total number of ways to assign closing days, ensuring no two bakeries close on the same day, is:

$$7 \text{ (choices for the first bakery)} \times 6 \text{ (choices for the second bakery)} \times 5 \text{ (choices for the third bakery)} \times 4 \text{ (choices for the fourth bakery)} = 7 \times 6 \times 5 \times 4$$

- Therefore, the correct answer is $7 \times 6 \times 5 \times 4$.

E UNORDERED DRAWS WITHOUT REPLACEMENT

E.1 SOLVING REAL-WORLD PROBLEMS

Ex 43: You're part of a sports squad with 5 players named P, Q, R, S, and T. How many different teams of 2 players can you form, where the order of players doesn't matter and no player is repeated?

$\boxed{10}$ teams

Answer: Let's break it down with two approaches:

- **Solution 1**: List the different teams of 2 players, where order doesn't matter and no repetition is allowed:

$$\{P,Q\},\{P,R\},\{P,S\},\{P,T\},\{Q,R\},$$
$$\{Q,S\},\{Q,T\},\{R,S\},\{R,T\},\{S,T\}$$

  There are 10 teams.

- **Solution 2**: This is a combination problem, as the order doesn't matter and no repetition is allowed. It's like drawing 2 balls without order and without replacement from a bag of 5 numbered balls (where each ball represents a player).

$$\begin{aligned}\binom{5}{2} &= \frac{5!}{2!(5-2)!}\\ &= \frac{5!}{2!\cdot 3!}\\ &= \frac{5\times 4\times \cancel{3\times 2\times 1}}{\cancel{3\times 2\times 1}\times 2\times 1}\\ &= \frac{5\times 4}{2\times 1}\\ &= 10\end{aligned}$$

  So, there are 10 different teams of 2 players that can be chosen from the squad.

Ex 44: You're part of a sports squad with 6 players named A, B, C, D, E, and F. How many different teams of 4 players can you form, where the order of players doesn't matter and no player is repeated?

$\boxed{15}$ teams

Answer: Let's figure it out with two approaches:

- **Solution 1**: List the different teams of 4 players, where order doesn't matter and no repetition is allowed:

$$\{A,B,C,D\},\{A,B,C,E\},\{A,B,C,F\},\{A,B,D,E\},$$
$$\{A,B,D,F\},\{A,B,E,F\},\{A,C,D,E\},\{A,C,D,F\},$$
$$\{A,C,E,F\},\{A,D,E,F\},\{B,C,D,E\},\{B,C,D,F\},$$
$$\{B,C,E,F\},\{B,D,E,F\},\{C,D,E,F\}$$

  There are 15 teams.

- **Solution 2**: This is a combination problem, as the order doesn't matter and no repetition is allowed. It's like drawing 4 balls without order and without replacement from a bag of 6 numbered balls (where each ball represents a player).

$$\begin{aligned}\binom{6}{4} &= \frac{6!}{4!(6-4)!}\\ &= \frac{6!}{4!\cdot 2!}\\ &= \frac{6\times 5\times \cancel{4\times 3\times 2\times 1}}{\cancel{4\times 3\times 2\times 1}\times 2\times 1}\\ &= \frac{6\times 5}{2\times 1}\\ &= 15\end{aligned}$$

  So, there are 15 different teams of 4 players that can be chosen from the squad.

Ex 45: Your company needs to form a committee of 4 members from a group of 9 employees. How many different teams of 4 can you choose, where the order of members doesn't matter and no employee is repeated? (Use a calculator to compute this.)

$\boxed{126}$ teams

Answer: Let's solve this:

- Forming a committee of 4 members from 9 employees, where order doesn't matter and no repetition is allowed, is a combination problem. It's like drawing 4 balls without order and without replacement from a bag of 9 numbered balls (where each ball represents an employee).

$$\begin{aligned}\binom{9}{4} &= \frac{9!}{4!(9-4)!}\\ &= \frac{9!}{4!\cdot 5!}\\ &= 126\end{aligned}$$

- So, there are 126 different teams of 4 that can be chosen from the group of 9 employees.

Ex 46: You're playing a word game and need to find the number of different anagrams of the word "TOTO" (an anagram is created by rearranging the letters of "TOTO," for example, "TTOO" is an anagram of "TOTO"). Use the analogy of drawing balls, where the number of the ball corresponds to the position of the letter T.

$\boxed{6}$ anagrams

Answer: Let's explore this with two approaches, using the ball-drawing analogy:

- **Solution 1**: List all the different anagrams of "TOTO," where the letters can be rearranged (noting there are 2 T's and 2 O's):

  TOTO, TTOO, OTTO, OTOT, OOTT, TOOT

  There are 6 anagrams.

- **Solution 2**: To find the number of anagrams, we count the number of ways to rearrange the letters of "TOTO." There are 4 positions (1, 2, 3, 4) for the letters, with 2 T's and 2 O's. Using the ball-drawing analogy, we draw 2 balls without order and without replacement from a bag of 4 numbered balls (where each ball number corresponds to a position for the letter T, and the remaining positions are filled with O's).

$$\begin{aligned}\binom{4}{2} &= \frac{4!}{2!(4-2)!} \\ &= \frac{4!}{2!\cdot 2!} \\ &= \frac{4\times 3\times \cancel{2\times 1}}{\cancel{2\times 1}\times 2\times 1} \\ &= \frac{4\times 3}{2\times 1} \\ &= 6\end{aligned}$$

So, there are 6 different anagrams of "TOTO," corresponding to the 6 ways to choose positions for the T's, with the remaining positions filled by O's.

Ex 47: You're playing a number game and need to find the number of different anagrams of the sequence "00111" (an anagram is created by rearranging the digits "00111," for example, "01011" is an anagram of "00111"). Use the analogy of drawing balls, where the number of the ball corresponds to the position of the digit 0.

$\boxed{10}$ anagrams

Answer: Let's explore this with two approaches, using the ball-drawing analogy:

- **Solution 1**: List all the different anagrams of "00111," where the digits can be rearranged (noting there are 2 zeros and 3 ones):

  00111, 01011, 01101, 01110, 10011, 10101, 10110, 11001, 11010, 11100

  There are 10 anagrams.

- **Solution 2**: To find the number of anagrams, we count the number of ways to rearrange the digits of "00111." There are 5 positions (1, 2, 3, 4, 5) for the digits, with 2 zeros and 3 ones. Using the ball-drawing analogy, we draw 2 balls without order and without replacement from a bag of 5 numbered balls (where each ball number corresponds to a position for the digit 0, and the remaining positions are filled with 1's).

$$\begin{aligned}\binom{5}{2} &= \frac{5!}{2!(5-2)!} \\ &= \frac{5!}{2!\cdot 3!} \\ &= \frac{5\times 4\times \cancel{3\times 2\times 1}}{\cancel{3\times 2\times 1}\times 2\times 1} \\ &= \frac{5\times 4}{2\times 1} \\ &= 10\end{aligned}$$

  So, there are 10 different anagrams of "00111," corresponding to the 10 ways to choose positions for the zeros, with the remaining positions filled by ones.

Ex 48: In your class of 15 students, there are 9 boys and 6 girls. The teacher wants to choose a group of 4 students, with exactly 2 boys and 2 girls. How many different ways can you form such a group? (Use a calculator to compute this.)

$\boxed{540}$ ways

Answer: Let's solve this:

- The teacher wants a group of 4 students with exactly 2 boys and 2 girls. This means we need to choose 2 boys from the 9 boys and 2 girls from the 6 girls, where order doesn't matter and no repetition is allowed. This is a combination problem, like drawing balls without order and without replacement from separate bags.
- Choose 2 boys out of 9:

$$\begin{aligned}\binom{9}{2} &= \frac{9!}{2!(9-2)!} \\ &= \frac{9!}{2!\cdot 7!} \\ &= \frac{9\times 8\times \cancel{7\times 6\times 5\times 4\times 3\times 2\times 1}}{\cancel{7\times 6\times 5\times 4\times 3\times 2\times 1}\times 2\times 1} \\ &= \frac{9\times 8}{2\times 1} \\ &= 36\end{aligned}$$

- Choose 2 girls out of 6:

$$\begin{aligned}\binom{6}{2} &= \frac{6!}{2!(6-2)!} \\ &= \frac{6!}{2!\cdot 4!} \\ &= \frac{6\times 5\times \cancel{4\times 3\times 2\times 1}}{\cancel{4\times 3\times 2\times 1}\times 2\times 1} \\ &= \frac{6\times 5}{2\times 1} \\ &= 15\end{aligned}$$

- Combine the selections: Since we need both 2 boys and 2 girls in the group, multiply the number of ways to choose the boys by the number of ways to choose the girls:

$$36\times 15 = 540$$

- So, there are 540 different ways to form a group of 4 students with exactly 2 boys and 2 girls.

Ex 49: You have a box with 10 balls, of which 6 are red and 4 are blue. How many different ways can you select 3 balls from the box such that exactly 2 of them are red? (Use a calculator to compute this.)

$\boxed{60}$ ways

Answer: Let's solve this:

- You need to select 3 balls, with exactly 2 red and 1 blue, where order doesn't matter and no repetition is allowed. This is a combination problem, like drawing balls without order and without replacement from separate bags for red and blue balls.
- Choose 2 red balls out of 6 red balls:

$$\begin{aligned}\binom{6}{2} &= \frac{6!}{2!(6-2)!} \\ &= \frac{6!}{2!\cdot 4!} \\ &= \frac{6\times 5\times \cancel{4\times 3\times 2\times 1}}{\cancel{4\times 3\times 2\times 1}\times 2\times 1} \\ &= \frac{6\times 5}{2\times 1} \\ &= 15\end{aligned}$$

- Choose 1 blue ball out of 4 blue balls:

$$\begin{aligned}\binom{4}{1} &= \frac{4!}{1!(4-1)!} \\ &= \frac{4!}{1!\cdot 3!} \\ &= \frac{4\times \cancel{3\times 2\times 1}}{\cancel{3\times 2\times 1}\times 1} \\ &= 4\end{aligned}$$

- Combine the selections: Multiply the number of ways to choose 2 red balls by the number of ways to choose 1 blue ball, since you need both in the group:

$$15\times 4 = 60$$

- So, there are 60 different ways to select 3 balls from the box with exactly 2 red and 1 blue.

Ex 50: Your company has 12 employees and wants to form a project team with 1 team leader, 1 deputy leader, and 4 team members. How many different ways can you form this project team, where the positions are distinct and no employee is repeated? (Use a calculator to compute this.)

$\boxed{27720}$ ways

Answer: Let's solve this:

- The team has distinct roles: 1 team leader, 1 deputy leader, and 4 team members, with no repetition allowed. This is a permutation problem, as the order (roles) matters, and it's like drawing balls in order, without replacement, from a bag of 12 numbered balls (where each ball represents an employee).
- Choose 1 team leader out of 12 employees:

$$\binom{12}{1} = \frac{12!}{1!(12-1)!} = 12$$

- Choose 1 deputy leader out of the remaining 11 employees:

$$\binom{11}{1} = \frac{11!}{1!(11-1)!} = 11$$

- Choose 4 team members out of the remaining 10 employees:

$$\begin{aligned}\binom{10}{4} &= \frac{10!}{4!(10-4)!} \\ &= \frac{10!}{4!\cdot 6!} \\ &= \frac{10\times 9\times 8\times 7\times \cancel{6\times 5\times 4\times 3\times 2\times 1}}{\cancel{6\times 5\times 4\times 3\times 2\times 1}\times 4\times 3\times 2\times 1} \\ &= \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} \\ &= \frac{5,040}{24} \\ &= 210\end{aligned}$$

- Combine the selections: Multiply the number of ways to choose each role, since the roles are distinct:

$$12\times 11\times 210 = 27,720$$

- So, there are 27,720 different ways to form the project team.

Ex 51: Your club has 10 members, and it wants to form a committee with 1 president, 1 secretary, and 3 members. How many different ways can you form this committee, where the positions are distinct and no member is repeated? (Use a calculator to compute this.)

$\boxed{5040}$ ways

Answer: Let's solve this:

- The committee has distinct roles: 1 president, 1 secretary, and 3 members, with no repetition allowed. This is a permutation problem, as the order (roles) matters, and it's like drawing balls in order, without replacement, from a bag of 10 numbered balls (where each ball represents a member).
- Choose 1 president out of 10 members:

$$\binom{10}{1} = \frac{10!}{1!(10-1)!} = 10$$

- Choose 1 secretary out of the remaining 9 members:

$$\binom{9}{1} = \frac{9!}{1!(9-1)!} = 9$$

- Choose 3 members out of the remaining 8 members:

$$\begin{aligned}\binom{8}{3} &= \frac{8!}{3!(8-3)!} \\ &= \frac{8!}{3!\cdot 5!} \\ &= \frac{8\times 7\times 6\times \cancel{5\times 4\times 3\times 2\times 1}}{\cancel{5\times 4\times 3\times 2\times 1}\times 3\times 2\times 1} \\ &= \frac{8\times 7\times 6}{3\times 2\times 1} \\ &= \frac{336}{6} \\ &= 56\end{aligned}$$

- Combine the selections: Multiply the number of ways to choose each role, since the roles are distinct:

$$10\times 9\times 56 = 5,040$$

- So, there are 5,040 different ways to form the committee.

E.2 COUNTING POKER HANDS

Ex 52: You're playing poker, and a hand consists of 5 cards drawn from a standard deck of 52 playing cards (13 ranks and 4 suits). How many different 5-card poker hands are possible? (Use a calculator to compute this.)

$\boxed{2598960}$ hands

Answer: Let's solve this:

- A 5-card poker hand is formed by choosing 5 cards from a deck of 52, where order doesn't matter and no card is repeated. This is a combination problem, like drawing 5 balls without order and without replacement from a bag of 52 numbered balls (where each ball represents a card).

$$\begin{aligned}\binom{52}{5} &= \frac{52!}{5!(52-5)!}\\ &= \frac{52!}{5!\cdot 47!}\\ &= 2,598,960 \quad \text{(enter 52!/(5! * 47!) in the calculator)}\end{aligned}$$

- So, there are 2,598,960 different possible 5-card poker hands.

Ex 53: In poker, a four-of-a-kind (also known as a square) is a hand where four cards have the same rank, and the fifth card is of a different rank. How many different four-of-a-kind hands are possible in a standard 52-card deck? (Use a calculator to compute this.)

$\boxed{624}$ hands

Answer: Let's break it down:

- To form a four-of-a-kind hand, choose 1 rank out of 13 for the four cards. This is like drawing 1 ball without order and without replacement from a bag of 13 numbered balls (where each ball represents a rank).

$$\binom{13}{1} = 13$$

- Choose all 4 cards of that rank from the 4 available suits. This is like drawing 4 balls without order and without replacement from a bag of 4 numbered balls (where each ball represents a suit).

$$\binom{4}{4} = 1$$

- Choose the fifth card from the remaining 48 cards (all cards not of the chosen rank). This is like drawing 1 ball without order and without replacement from a bag of 48 numbered balls (where each ball represents a card).

$$\binom{48}{1} = 48$$

- Combine the selections: Multiply the number of ways to choose each part, since each choice is independent:

$$13 \times 1 \times 48 = 624$$

- So, there are 624 different four-of-a-kind hands possible in a standard 52-card deck.

Ex 54: In poker, a full house is a hand that contains three cards of one rank and two cards of another rank. How many different full house hands are possible in a standard 52-card deck? (Use a calculator to compute this.)

$\boxed{3744}$ hands

Answer: Let's figure it out:

- To form a full house, choose 1 rank out of 13 for the three cards. This is like drawing 1 ball without order and without replacement from a bag of 13 numbered balls (where each ball represents a rank).

$$\binom{13}{1} = 13$$

- Choose 3 cards of that rank from the 4 available suits. This is like drawing 3 balls without order and without replacement from a bag of 4 numbered balls (where each ball represents a suit).

$$\binom{4}{3} = 4$$

- Choose another rank (different from the first) out of the remaining 12 ranks for the two cards. This is like drawing 1 ball without order and without replacement from a bag of 12 numbered balls (where each ball represents a rank).

$$\binom{12}{1} = 12$$

- Choose 2 cards of that second rank from the 4 available suits. This is like drawing 2 balls without order and without replacement from a bag of 4 numbered balls (where each ball represents a suit).

$$\binom{4}{2} = 6$$

- Combine the selections: Multiply the number of ways to choose each part, since each choice is independent:

$$13 \times 4 \times 12 \times 6 = 3,744$$

- So, there are 3,744 different full house hands possible in a standard 52-card deck.

Ex 55: In poker, a three of a kind is a hand that contains three cards of one rank and two other cards of different ranks, which are not the same as each other or the same as the rank of the three cards. How many different three-of-a-kind hands are possible in a standard 52-card deck? (Use a calculator to compute this.)

$\boxed{54912}$ hands

Answer: Let's solve this:

- To form a three-of-a-kind hand, choose 1 rank out of 13 for the three cards. This is like drawing 1 ball without order and without replacement from a bag of 13 numbered balls (where each ball represents a rank).

$$\binom{13}{1} = 13$$

- Choose 3 cards of that rank from the 4 available suits. This is like drawing 3 balls without order and without replacement from a bag of 4 numbered balls (where each ball represents a suit).

$$\binom{4}{3} = 4$$

- Choose 2 different ranks (neither the same as each other nor the same as the three-of-a-kind rank) out of the remaining 12 ranks for the two other cards. This is like drawing 2 balls without order and without replacement from a bag of 12 numbered balls (where each ball represents a rank).

$$\binom{12}{2} = \frac{12!}{2!(12-2)!} = \frac{12\times 11}{2\times 1} = 66$$

- Choose 1 card of the first rank from the 4 available suits, and 1 card of the second rank from the 4 available suits. This is like drawing 1 ball without order and without replacement from a bag of 4 numbered balls for each rank (where each ball represents a suit).

$$\binom{4}{1} \times \binom{4}{1} = 4 \times 4 = 16$$

- Combine the selections: Multiply the number of ways to choose each part, since each choice is independent:

$$13 \times 4 \times 66 \times 16 = 54,912$$

- So, there are 54,912 different three-of-a-kind hands possible in a standard 52-card deck.

Ex 56: In poker, a two pair is a hand that contains two cards of one rank, two cards of another rank, and one card of a different rank from the other four cards. How many different two-pair hands are possible in a standard 52-card deck? (Use a calculator to compute this.)

$\boxed{123552}$ hands

Answer: Let's work through it:

- To form a two-pair hand, choose 2 different ranks out of 13 for the pairs. This is like drawing 2 balls without order and without replacement from a bag of 13 numbered balls (where each ball represents a rank).

$$\binom{13}{2} = \frac{13!}{2!(13-2)!} = \frac{13 \times 12}{2 \times 1} = 78$$

- Choose 2 cards of the first rank from the 4 available suits. This is like drawing 2 balls without order and without replacement from a bag of 4 numbered balls (where each ball represents a suit).

$$\binom{4}{2} = 6$$

- Choose 2 cards of the second rank from the 4 available suits. This is like drawing 2 balls without order and without replacement from a bag of 4 numbered balls (where each ball represents a suit).

$$\binom{4}{2} = 6$$

- Choose a third rank (different from the two pairs) out of the remaining 11 ranks for the fifth card. This is like drawing 1 ball without order and without replacement from a bag of 11 numbered balls (where each ball represents a rank).

$$\binom{11}{1} = 11$$

- Choose 1 card of that third rank from the 4 available suits. This is like drawing 1 ball without order and without replacement from a bag of 4 numbered balls (where each ball represents a suit).

$$\binom{4}{1} = 4$$

- Combine the selections: Multiply the number of ways to choose each part, since each choice is independent:

$$78 \times 6 \times 6 \times 11 \times 4 = 123,552$$

- So, there are 123,552 different two-pair hands possible in a standard 52-card deck.

Ex 57: In poker, a one pair is a hand that contains two cards of one rank and three other cards of different ranks, which are not the same as each other or the same as the rank of the pair. How many different one-pair hands are possible in a standard 52-card deck? (Use a calculator to compute this.)

$\boxed{1098240}$ hands

Answer: Let's work through it:

- To form a one-pair hand, choose 1 rank out of 13 for the pair. This is like drawing 1 ball without order and without replacement from a bag of 13 numbered balls (where each ball represents a rank).

$$\binom{13}{1} = 13$$

- Choose 2 cards of that rank from the 4 available suits. This is like drawing 2 balls without order and without replacement from a bag of 4 numbered balls (where each ball represents a suit).

$$\binom{4}{2} = 6$$

- Choose 3 different ranks (none the same as each other or the pair's rank) out of the remaining 12 ranks for the three other cards. This is like drawing 3 balls without order and without replacement from a bag of 12 numbered balls (where each ball represents a rank).

$$\binom{12}{3} = \frac{12!}{3!(12-3)!} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220$$

- Choose 1 card of each of the 3 chosen ranks from the 4 available suits for each rank. This is like drawing 1 ball without order and without replacement from a bag of 4 numbered balls for each rank (where each ball represents a suit).

$$\binom{4}{1}^3 = 4 \times 4 \times 4 = 64$$

- Combine the selections: Multiply the number of ways to choose each part, since each choice is independent:

$$13 \times 6 \times 220 \times 64 = 1,098,240$$

- So, there are 1 098 240 different one-pair hands possible in a standard 52-card deck.