CONTINUOUS RANDOM VARIABLES

A DEFINITIONS

A.1 CONTINUOUS RANDOM VARIABLE

A.1.1 DISTINGUISHING BETWEEN DISCRETE AND CONTINUOUS VARIABLES

MCQ 1: Determine whether the following random variable is discrete or continuous.
The number of heads obtained after flipping a coin 10 times.

☒ Discrete

☐ Continuous

Answer: The number of heads can only take on a finite number of specific values: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Since we can list all possible outcomes, the variable is **discrete**.

MCQ 2: Determine whether the following random variable is discrete or continuous.
The height of a randomly selected student in a school.

☐ Discrete

☒ Continuous

Answer: Height can take any value within a certain range (e.g., between 150 cm and 190 cm). It is not restricted to a countable number of specific values. Therefore, the variable is **continuous**.

MCQ 3: Determine whether the following random variable is discrete or continuous.
The number of cars that pass through a certain intersection in one hour.

☒ Discrete

☐ Continuous

Answer: The number of cars can be counted using whole numbers: 0, 1, 2, 3, ... While the number could be very large, it is countable. Therefore, the variable is **discrete**.

MCQ 4: Determine whether the following random variable is discrete or continuous.
The time it takes for a student to complete a 100-meter race.

☐ Discrete

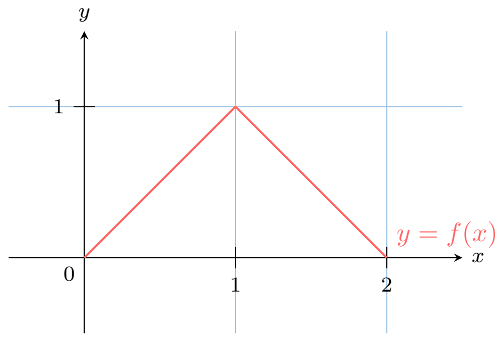
☒ Continuous

Answer: Time is a measurement that can take any value within an interval (e.g., between 11.5 seconds and 12.5 seconds). It is not limited to specific, countable values. Therefore, the variable is **continuous**.

A.2 PROBABILITY DENSITY FUNCTION

A.2.1 CALCULATING PROBABILITIES UNDER THE CURVE

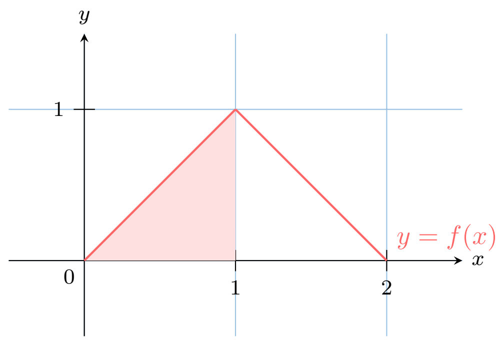
Ex 5: Suppose X represents the time (in hours) a device operates before needing maintenance, with values on $[0, 2]$, and its probability density function $f(x)$ is shown in the graph below.



Using the graph, estimate the probability that the device operates for 1 hour or less.

$$P(0 \leq X \leq 1) = \boxed{\frac{1}{2}}$$

Answer: Consider the graph of $f(x)$, which forms a triangle over $[0, 2]$:



The probability $P(0 \leq X \leq 1)$ is the area under $f(x)$ from 0 to 1, shaded in the figure. This forms a right triangle with:

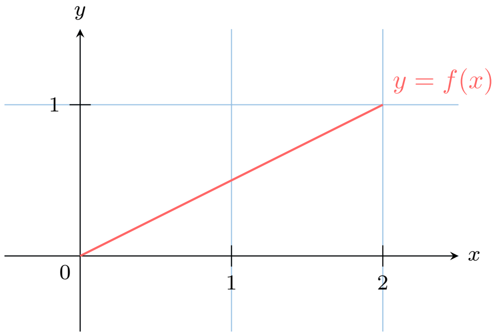
- Base $= 1$ (from 0 to 1),
- Height $= 1$ (at $x = 1, f(x) = 1$).

Thus:

$$\begin{aligned} P(0 \leq X \leq 1) &= \int_0^1 f(x)\,dx \\ &= \text{area of the shaded triangle} \\ &= \frac{\text{base} \times \text{height}}{2} \\ &= \frac{1 \times 1}{2} \\ &= \frac{1}{2} \end{aligned}$$

So the probability is $\frac{1}{2}$.

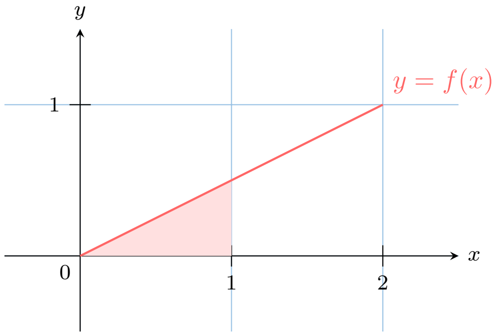
Ex 6: Suppose X represents the waiting time (in minutes) for a bus, with values on $[0, 2]$, and its probability density function is shown in the graph below.



Using the graph, estimate the probability that the waiting time is less than or equal to 1 minute.

$$P(X \leq 1) = \boxed{\frac{1}{4}}$$

Answer: Consider the graph of $f(x) = \frac{x}{2}$, which increases linearly from 0 to 1 over $[0, 2]$:



The probability $P(X \leq 1)$ is the area under $f(x)$ from 0 to 1, shaded in the figure. This forms a right triangle with:

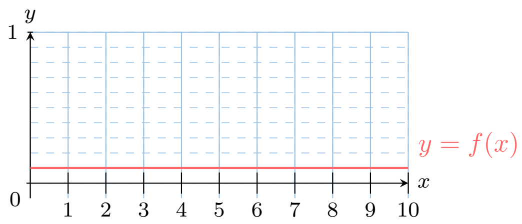
- Base $= 1$ (from 0 to 1),
- Height $= f(1) = \frac{1}{2}$ (at $x = 1$).

Thus:

$$\begin{aligned} P(X \leq 1) &= \int_0^1 f(x)\,dx \\ &= \int_0^1 \frac{x}{2}\,dx \\ &= \text{area of the shaded triangle} \\ &= \frac{\text{base} \times \text{height}}{2} \\ &= \frac{1 \times \frac{1}{2}}{2} \\ &= \frac{\frac{1}{2}}{2} \\ &= \frac{1}{4} \end{aligned}$$

So the probability is $\frac{1}{4}$.

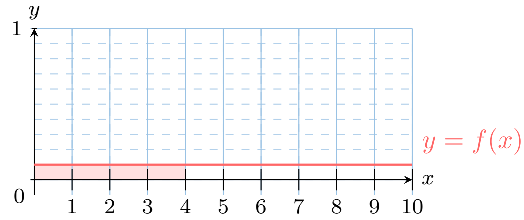
Ex 7: Suppose X represents the waiting time (in minutes) for a bus, which follows a uniform distribution over $[0, 10]$, and its probability density function $f(x)$ is shown in the graph below.



Using the graph, estimate the probability that the waiting time is 4 minutes or less.

$$P(0 \leq X \leq 4) = \boxed{\frac{2}{5}}$$

Answer: Consider the graph of $f(x)$, which is constant at $f(x) = \frac{1}{10}$ over $[0, 10]$:



The probability $P(0 \leq X \leq 4)$ is the area under $f(x)$ from 0 to 4, shaded in the figure. This forms a rectangle with:

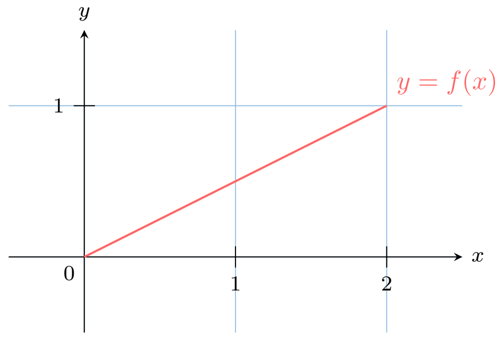
- Width $= 4$ (from 0 to 4),
- Height $= \frac{1}{10}$ (the value of $f(x)$ for a uniform distribution over 10 minutes).

Thus:

$$\begin{aligned} P(0 \leq X \leq 4) &= \int_0^4 f(x)\,dx \\ &= \text{area of the shaded rectangle} \\ &= \text{width} \times \text{height} \\ &= 4 \times \frac{1}{10} \\ &= \frac{4}{10} \\ &= \frac{2}{5} \end{aligned}$$

So the probability is $\frac{2}{5}$.

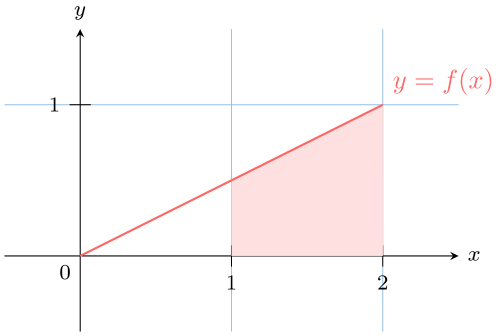
Ex 8: Suppose X represents the waiting time (in minutes) for a bus, with values on $[0, 2]$, and its probability density function is given by $f(x) = \frac{x}{2}$, as shown in the graph below.



Using the graph, estimate the probability that the waiting time is more than 1 minute.

$$P(X > 1) = \boxed{\frac{3}{4}}$$

Answer: Consider the graph of $f(x) = \frac{x}{2}$, which increases linearly from 0 to 1 over $[0, 2]$:



The probability $P(X > 1)$ is the area under $f(x)$ from 1 to 2, shaded in the figure. This forms a trapezoid with:

- Base 1 $= f(1) = \frac{1}{2} = 0.5$ (at $x = 1$),
- Base 2 $= f(2) = \frac{2}{2} = 1$ (at $x = 2$),
- Height $= 1$ (from 1 to 2).

Alternatively, since the total area under $f(x)$ from 0 to 2 is 1, and $P(X \leq 1) = \frac{1}{4}$ (from the previous exercise), we can use the complement:

$$\begin{aligned} P(X > 1) &= 1 - P(X \leq 1) \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

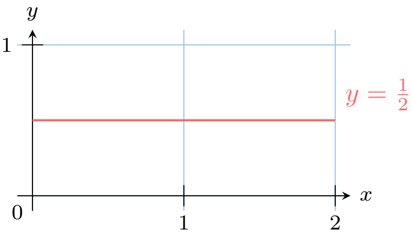
To confirm geometrically, the trapezoid area is:

$$\begin{aligned} P(X > 1) &= \int_1^2 \frac{x}{2}\,dx \\ &= \text{area of the shaded trapezoid} \\ &= \frac{\text{base 1} + \text{base 2}}{2} \times \text{height} \\ &= \frac{0.5 + 1}{2} \times 1 \\ &= \frac{1.5}{2} \\ &= \frac{3}{4} \end{aligned}$$

So the probability is $\frac{3}{4}$.

A.2.2 VERIFYING THAT $f(x)$ IS A PROBABILITY DENSITY FUNCTION

Ex 9: Consider the function $f(x) = \frac{1}{2}$, defined on the interval $[0, 2]$.



Verify that $f(x)$ is a probability density function on the interval $[0, 2]$.

Answer:

- **Check if $f(x) \geq 0$:**
  - *Graphical Approach:* From the graph, we observe that $f(x) = \frac{1}{2}$ is non-negative over $[0, 2]$ (graph above the x-axis).
  - *Algebraic Approach:* Let $x \in [0, 2]$.

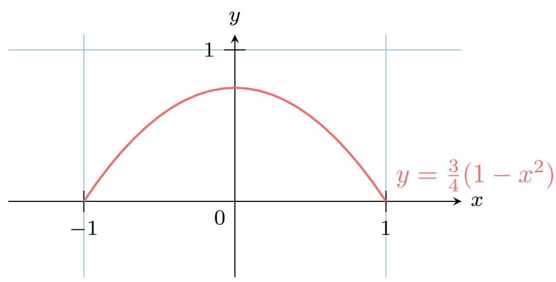
$$\begin{aligned} \frac{1}{2} &\geq 0 \\ f(x) &\geq 0 \end{aligned}$$

- **Check if the total area equals 1:**

$$\begin{aligned} \int_0^2 f(x)\,dx &= \int_0^2 \frac{1}{2}\,dx \\ &= \frac{1}{2}\int_0^2 1\,dx \\ &= \frac{1}{2}[x]_0^2 \\ &= \frac{1}{2}[2 - 0] \\ &= \frac{1}{2}\cdot 2 \\ &= 1 \end{aligned}$$

Since both conditions are satisfied—$f(x) \geq 0$ on $[0, 2]$ and $\int_0^2 f(x)\,dx = 1$—we conclude that $f(x)$ is indeed a probability density function on the interval $[0, 2]$.

Ex 10: Consider the function $f(x) = \frac{3}{4}(1 - x^2)$, defined on the interval $[-1, 1]$.



Verify that $f(x)$ is a probability density function on the interval $[-1, 1]$.

Answer:

- **Check if $f(x) \geq 0$:**
  - *Graphical Approach:* From the graph, we observe that $f(x) = \frac{3}{4}(1 - x^2)$ is non-negative over $[-1, 1]$ (graph above the x-axis).
  - *Algebraic Approach:* Let $x \in [-1, 1]$.

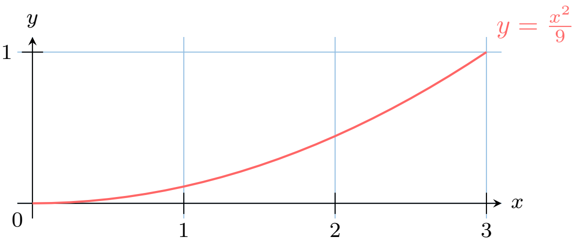
$$\begin{aligned} -1 &\leq x \leq 1 \\ 0 &\leq x^2 \leq 1 \\ -1 &\leq -x^2 \leq 0 \\ 0 &\leq 1 - x^2 \leq 1 \\ 0 &\leq \frac{3}{4}(1 - x^2) \\ 0 &\leq f(x) \end{aligned}$$

- **Check if the total area equals 1:**

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx &= \int_{-1}^{1} \frac{3}{4}(1-x^2)\,dx \\ &= \frac{3}{4}\int_{-1}^{1}(1-x^2)\,dx \\ &= \frac{3}{4}\left[x-\frac{x^3}{3}\right]_{-1}^{1} \\ &= \frac{3}{4}\left[\left(1-\frac{1^3}{3}\right)-\left((-1)-\frac{(-1)^3}{3}\right)\right] \\ &= \frac{3}{4}\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right] \\ &= \frac{3}{4}\left[\frac{2}{3}+\frac{2}{3}\right] \\ &= \frac{3}{4}\cdot\frac{4}{3} \\ &= 1\end{aligned}$$

Since both conditions are satisfied—$f(x) \geq 0$ on $[-1,1]$ and $\int_{-1}^{1} f(x)\,dx = 1$—we conclude that $f(x)$ is indeed a probability density function on the interval $[-1,1]$.

Ex 11: Consider the function $f(x) = \frac{x^2}{9}$, defined on the interval $[0,3]$.



Verify that $f(x)$ is a probability density function on the interval $[0,3]$.

Answer:

- **Check if $f(x) \geq 0$:**
  - *Graphical Approach:* From the graph, we observe that $f(x) = \frac{x^2}{9}$ is non-negative over $[0,3]$ (graph above the x-axis).
  - *Algebraic Approach:* Let $x \in [0,3]$.

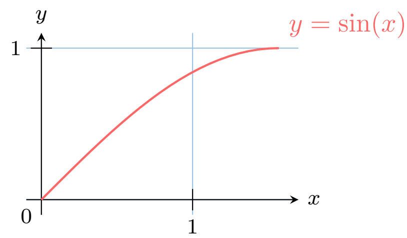
$$\begin{aligned}0 &\leq x^2 \\ 0 &\leq \frac{x^2}{9} \\ 0 &\leq f(x)\end{aligned}$$

- **Check if the total area equals 1:**

$$\begin{aligned}\int_{0}^{3} f(x)\,dx &= \int_{0}^{3} \frac{x^2}{9}\,dx \\ &= \frac{1}{9}\int_{0}^{3} x^2\,dx \\ &= \frac{1}{9}\left[\frac{x^3}{3}\right]_{0}^{3} \\ &= \frac{1}{9}\left[\frac{3^3}{3}-\frac{0^3}{3}\right] \\ &= \frac{1}{9}\left[\frac{27}{3}-0\right] \\ &= \frac{1}{9}\cdot 9 \\ &= 1\end{aligned}$$

Since both conditions are satisfied—$f(x) \geq 0$ on $[0,3]$ and $\int_{0}^{3} f(x)\,dx = 1$—we conclude that $f(x)$ is indeed a probability density function on the interval $[0,3]$.

Ex 12: Consider the function $f(x) = \sin(x)$, defined on the interval $[0, \frac{\pi}{2}]$.



Verify that $f(x)$ is a probability density function on the interval $[0, \frac{\pi}{2}]$.

Answer:

- **Check if $f(x) \geq 0$:**
  - *Graphical Approach:* From the graph, we observe that $f(x) = \sin(x)$ is non-negative over $[0, \frac{\pi}{2}]$ (graph above the x-axis).
  - *Algebraic Approach:* Let $x \in [0, \frac{\pi}{2}]$.

$$\begin{aligned}0 &\leq \sin(x) \\ 0 &\leq f(x)\end{aligned}$$

- **Check if the total area equals 1:**

$$\begin{aligned}\int_{0}^{\frac{\pi}{2}} f(x)\,dx &= \int_{0}^{\frac{\pi}{2}} \sin(x)\,dx \\ &= [-\cos(x)]_{0}^{\frac{\pi}{2}} \\ &= -\cos\left(\frac{\pi}{2}\right)-(-\cos(0)) \\ &= 0+1 \\ &= 1\end{aligned}$$

Since both conditions are satisfied—$f(x) \geq 0$ on $[0, \frac{\pi}{2}]$ and $\int_{0}^{\frac{\pi}{2}} f(x)\,dx = 1$—we conclude that $f(x)$ is indeed a probability density function on the interval $[0, \frac{\pi}{2}]$.

A.2.3 NORMALIZING A PROBABILITY DENSITY FUNCTION

Ex 13: Consider the function $f(x) = a$.
Find the value of a such that $f(x)$ is a probability density function on the interval $[0, 2]$.

Answer:

- **Check if the total area equals 1:**

$$\begin{aligned}
1 &= \int_0^2 f(x)\,dx \\
1 &= \int_0^2 a\,dx \\
1 &= a\int_0^2 1\,dx \\
1 &= a\,[x]_0^2 \\
1 &= a\,(2-0) \\
1 &= 2a \\
a &= \frac{1}{2}
\end{aligned}$$

- **Check $f(x) \geq 0$ with $a = \frac{1}{2}$:**

$$f(x) = \frac{1}{2}$$

Since $\frac{1}{2} > 0$ for all $x \in [0, 2]$, we have:

$$f(x) = \frac{1}{2} \geq 0$$

Thus, for $a = \frac{1}{2}$, $f(x) = \frac{1}{2}$ is a probability density function on the interval $[0, 2]$.

Ex 14: Consider the function $f(x) = ax^3$.
Find the value of a such that $f(x)$ is a probability density function on the interval $[0, 2]$.

Answer:

- **Check if the total area equals 1:**

$$\begin{aligned}
1 &= \int_0^2 f(x)\,dx \\
1 &= \int_0^2 ax^3\,dx \\
1 &= a\int_0^2 x^3\,dx \\
1 &= a\left[\frac{x^4}{4}\right]_0^2 \\
1 &= a\left(\frac{2^4}{4} - \frac{0^4}{4}\right) \\
1 &= a\left(\frac{16}{4} - 0\right) \\
1 &= a \cdot 4 \\
1 &= 4a \\
a &= \frac{1}{4}
\end{aligned}$$

- **Check $f(x) \geq 0$ with $a = \frac{1}{4}$:**

$$f(x) = \frac{1}{4}x^3 \geq 0$$

Thus, for $a = \frac{1}{4}$, $f(x) = \frac{1}{4}x^3$ is a probability density function on the interval $[0, 2]$.

Ex 15: Consider the function $f(x) = a\frac{1}{x}$.
Find the value of a such that $f(x)$ is a probability density function on the interval $[1, 2]$.

Answer:

- **Check if the total area equals 1:**

$$\begin{aligned}
1 &= \int_1^2 f(x)\,dx \\
1 &= \int_1^2 a\frac{1}{x}\,dx \\
1 &= a\int_1^2 \frac{1}{x}\,dx \\
1 &= a\,[\ln(x)]_1^2 \\
1 &= a\,(\ln(2) - \ln(1)) \\
1 &= a\,(\ln(2) - 0) \\
1 &= a\ln(2) \\
a &= \frac{1}{\ln(2)}
\end{aligned}$$

- **Check $f(x) \geq 0$ with $a = \frac{1}{\ln(2)}$:**

$$f(x) = \frac{1}{\ln(2)} \cdot \frac{1}{x}$$

Since $\ln(2) > 0$ and $\frac{1}{x} > 0$ for $x \in [1, 2]$, we have:

$$f(x) = \frac{1}{\ln(2)} \cdot \frac{1}{x} \geq 0$$

Thus, for $a = \frac{1}{\ln(2)}$, $f(x) = \frac{1}{\ln(2)} \cdot \frac{1}{x}$ is a probability density function on the interval $[1, 2]$.

Ex 16: Consider the function $f(x) = a\sqrt{x}$.
Find the value of a such that $f(x)$ is a probability density function on the interval $[0, 4]$.

Answer:

- **Check if the total area equals 1:**

$$\begin{aligned}
1 &= \int_0^4 f(x)\,dx \\
1 &= \int_0^4 a\sqrt{x}\,dx \\
1 &= a\int_0^4 x^{\frac{1}{2}}\,dx \\
1 &= a\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 \\
1 &= a \cdot \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^4 \\
1 &= a \cdot \frac{2}{3}\left(4^{\frac{3}{2}} - 0\right) \\
1 &= a \cdot \frac{2}{3} \cdot 8 \\
1 &= a \cdot \frac{16}{3} \\
a &= \frac{3}{16}
\end{aligned}$$

- **Check $f(x) \geq 0$ with $a = \frac{3}{16}$:**

$$f(x) = \frac{3}{16}\sqrt{x}$$

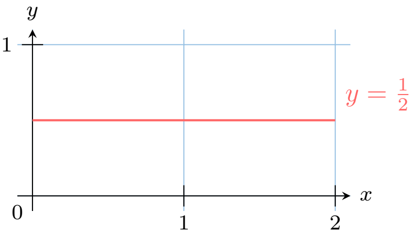
Since $\frac{3}{16} > 0$ and $\sqrt{x} \geq 0$ for $x \in [0, 4]$, we have:

$$f(x) = \frac{3}{16}\sqrt{x} \geq 0$$

Thus, for $a = \frac{3}{16}$, $f(x) = \frac{3}{16}\sqrt{x}$ is a probability density function on the interval $[0, 4]$.

A.2.4 FINDING A PROBABILITY

Ex 17: The random variable X has the density $f(x) = \frac{1}{2}$, on the interval $[0, 2]$.

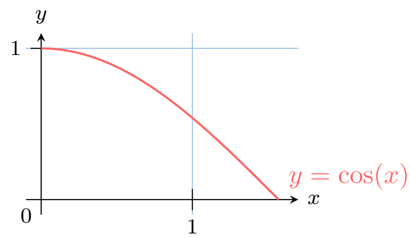


Find $P(\frac{1}{2} \leq X \leq \frac{3}{4})$.

Answer:

$$\begin{aligned}
P\left(\frac{1}{2} \leq X \leq \frac{3}{4}\right) &= \int_{\frac{1}{2}}^{\frac{3}{4}} f(x)\, \mathrm{d}x \\
&= \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{1}{2}\, \mathrm{d}x \\
&= \frac{1}{2}\,[x]_{\frac{1}{2}}^{\frac{3}{4}} \\
&= \frac{1}{2}\left[\frac{3}{4} - \frac{1}{2}\right] \\
&= \frac{1}{2}\left[\frac{3}{4} - \frac{2}{4}\right] \\
&= \frac{1}{2} \cdot \frac{1}{4} \\
&= \frac{1}{8}
\end{aligned}$$

Ex 18: The random variable X has the density $f(x) = \cos(x)$, on the interval $[0, \frac{\pi}{2}]$.



Find $P(0 \leq X \leq \frac{\pi}{4})$.

Answer:

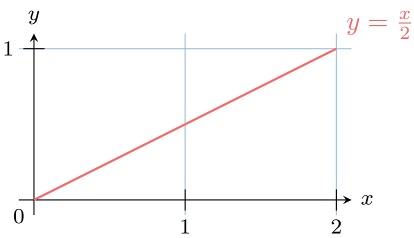
$$\begin{aligned}
P\left(0 \leq X \leq \frac{\pi}{4}\right) &= \int_0^{\frac{\pi}{4}} f(x)\, \mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \cos(x)\, \mathrm{d}x \\
&= [\sin(x)]_0^{\frac{\pi}{4}} \\
&= \sin\left(\frac{\pi}{4}\right) - \sin(0) \\
&= \frac{\sqrt{2}}{2} - 0 \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

Ex 19: The random variable X has the density $f(x) = \frac{1}{\ln(2)x}$, on the interval $[1, 2]$.
Find $P\left(1 \leq X \leq \frac{3}{2}\right)$.

Answer:

$$\begin{aligned}
P\left(1 \leq X \leq \frac{3}{2}\right) &= \int_1^{\frac{3}{2}} f(x)\, \mathrm{d}x \\
&= \int_1^{\frac{3}{2}} \frac{1}{\ln(2)x}\, \mathrm{d}x \\
&= \frac{1}{\ln 2} \int_1^{\frac{3}{2}} \frac{1}{x}\, \mathrm{d}x \\
&= \frac{1}{\ln 2}\,[\ln x]_1^{\frac{3}{2}} \\
&= \frac{1}{\ln 2}\left(\ln\frac{3}{2} - \ln 1\right) \\
&= \frac{1}{\ln 2}\left(\ln\frac{3}{2} - 0\right) \\
&= \frac{\ln(3/2)}{\ln 2}
\end{aligned}$$

Ex 20: The random variable X has the density $f(x) = \frac{x}{2}$, on the interval $[0, 2]$.



Find $P(1 \leq X \leq 2)$.

Answer:

$$\begin{aligned}
P(1 \leq X \leq 2) &= \int_1^2 f(x)\,\mathrm{d}x \\
&= \int_1^2 \frac{x}{2}\,\mathrm{d}x \\
&= \frac{1}{2}\int_1^2 x\,\mathrm{d}x \\
&= \frac{1}{2}\left[\frac{x^2}{2}\right]_1^2 \\
&= \frac{1}{2}\left(\frac{2^2}{2}-\frac{1^2}{2}\right) \\
&= \frac{1}{2}\left(2-\frac{1}{2}\right) \\
&= \frac{1}{2}\cdot\frac{3}{2} \\
&= \frac{3}{4}
\end{aligned}$$

A.3 EXPECTATION

A.3.1 CALCULATING AN EXPECTATION

Ex 21: The random variable X has the density $f(x) = \frac{1}{2}$, on the interval $[0, 2]$. Calculate $E(X)$.

Answer:

$$\begin{aligned}
E(X) &= \int_a^b x f(x)\,dx \\
&= \int_0^2 x\frac{1}{2}\,dx \\
&= \frac{1}{2}\int_0^2 x\,dx \\
&= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 \\
&= \frac{1}{2}\left[\frac{2^2}{2}-\frac{0^2}{2}\right] \\
&= \frac{1}{2}\cdot\frac{4}{2} \\
&= 1
\end{aligned}$$

Ex 22: The random variable X has the density $f(x) = \frac{1}{\ln(2)x}$, on the interval $[1, 2]$. Calculate $E(X)$.

Answer:

$$\begin{aligned}
E(X) &= \int_a^b x f(x)\,dx \\
&= \int_1^2 x\frac{1}{\ln(2)x}\,dx \\
&= \frac{1}{\ln 2}\int_1^2 1\,dx \\
&= \frac{1}{\ln 2}[x]_1^2 \\
&= \frac{1}{\ln 2}(2-1) \\
&= \frac{1}{\ln 2}
\end{aligned}$$

Ex 23: The random variable X has the density $f(x) = \frac{x}{2}$, on the interval $[0, 2]$. Calculate $E(X)$.

Answer:

$$\begin{aligned}
E(X) &= \int_a^b x f(x)\,dx \\
&= \int_0^2 x\frac{x}{2}\,dx \\
&= \frac{1}{2}\int_0^2 x^2\,dx \\
&= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 \\
&= \frac{1}{2}\left(\frac{2^3}{3}-\frac{0^3}{3}\right) \\
&= \frac{1}{2}\left(\frac{8}{3}-0\right) \\
&= \frac{1}{2}\cdot\frac{8}{3} \\
&= \frac{4}{3}
\end{aligned}$$

Ex 24: The random variable X has the density $f(x) = \frac{2}{x^2}$, on the interval $[1, 2]$. Calculate $E(X)$.

Answer:

$$\begin{aligned}
E(X) &= \int_a^b x f(x)\,dx \\
&= \int_1^2 x\frac{2}{x^2}\,dx \\
&= \int_1^2 \frac{2}{x}\,dx \\
&= 2\int_1^2 \frac{1}{x}\,dx \\
&= 2\,[\ln x]_1^2 \\
&= 2\,(\ln 2 - \ln 1) \\
&= 2\,(\ln 2 - 0) \\
&= 2\ln 2
\end{aligned}$$

A.4 VARIANCE

A.4.1 CALCULATING A VARIANCE

Ex 25: The random variable X with values on $[-1, 1]$ has density $f(x) = \frac{1}{2}$.
Find $V(X)$.

Answer:

- Compute $E(X)$:

$$\begin{aligned}
E(X) &= \int_{-1}^1 x\cdot\frac{1}{2}\,dx \\
&= \frac{1}{2}\int_{-1}^1 x\,dx \\
&= \frac{1}{2}\left[\frac{x^2}{2}\right]_{-1}^1 \\
&= \frac{1}{2}\left(\frac{1^2}{2}-\frac{(-1)^2}{2}\right) \\
&= \frac{1}{2}\left(\frac{1}{2}-\frac{1}{2}\right) \\
&= \frac{1}{2}\cdot 0 \\
&= 0
\end{aligned}$$

- Compute $\int_{-1}^{1} x^2 \cdot f(x)\,dx$:

$$\begin{aligned}
\int_{-1}^{1} x^2 \cdot f(x)\,dx &= \int_{-1}^{1} x^2 \cdot \frac{1}{2}\,dx \\
&= \frac{1}{2}\int_{-1}^{1} x^2\,dx \\
&= \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} \\
&= \frac{1}{2}\left(\frac{1^3}{3} - \frac{(-1)^3}{3}\right) \\
&= \frac{1}{2}\left(\frac{1}{3} - \left(-\frac{1}{3}\right)\right) \\
&= \frac{1}{2}\left(\frac{1}{3} + \frac{1}{3}\right) \\
&= \frac{1}{2} \cdot \frac{2}{3} \\
&= \frac{1}{3}
\end{aligned}$$

- Compute $V(X)$ using the alternative formula:

$$\begin{aligned}
V(X) &= \int_{-1}^{1} x^2 \cdot f(x)\,dx - [E(X)]^2 \\
&= \frac{1}{3} - (0)^2 \\
&= \frac{1}{3}
\end{aligned}$$

Ex 26: The random variable X with values on $[0, 2]$ has density $f(x) = \frac{x}{2}$.
Find $V(X)$.

Answer:

- Compute $E(X)$:

$$\begin{aligned}
E(X) &= \int_{0}^{2} x \cdot \frac{x}{2}\,dx \\
&= \int_{0}^{2} \frac{x^2}{2}\,dx \\
&= \left[\frac{x^3}{6}\right]_{0}^{2} \\
&= \frac{2^3}{6} - 0 \\
&= \frac{8}{6} \\
&= \frac{4}{3}
\end{aligned}$$

- Compute $\int_{0}^{2} x^2 \cdot f(x)\,dx$:

$$\begin{aligned}
\int_{0}^{2} x^2 \cdot f(x)\,dx &= \int_{0}^{2} x^2 \cdot \frac{x}{2}\,dx \\
&= \int_{0}^{2} \frac{x^3}{2}\,dx \\
&= \left[\frac{x^4}{8}\right]_{0}^{2} \\
&= \frac{2^4}{8} - 0 \\
&= \frac{16}{8} \\
&= 2
\end{aligned}$$

- Compute $V(X)$ using the alternative formula:

$$\begin{aligned}
V(X) &= \int_{0}^{2} x^2 \cdot f(x)\,dx - [E(X)]^2 \\
&= 2 - \left(\frac{4}{3}\right)^2 \\
&= 2 - \frac{16}{9} \\
&= \frac{18}{9} - \frac{16}{9} \\
&= \frac{2}{9}
\end{aligned}$$

Ex 27: The random variable X with values on $[1, 2]$ has density $f(x) = \frac{2}{x^2}$.
Find $V(X)$.

Answer:

- Compute $E(X)$:

$$\begin{aligned}
E(X) &= \int_{1}^{2} x \cdot \frac{2}{x^2}\,dx \\
&= \int_{1}^{2} \frac{2}{x}\,dx \\
&= 2\int_{1}^{2} \frac{1}{x}\,dx \\
&= 2\left[\ln x\right]_{1}^{2} \\
&= 2(\ln 2 - \ln 1) \\
&= 2(\ln 2 - 0) \\
&= 2\ln 2
\end{aligned}$$

- Compute $\int_{1}^{2} x^2 \cdot f(x)\,dx$:

$$\begin{aligned}
\int_{1}^{2} x^2 \cdot f(x)\,dx &= \int_{1}^{2} x^2 \cdot \frac{2}{x^2}\,dx \\
&= \int_{1}^{2} 2\,dx \\
&= 2\left[x\right]_{1}^{2} \\
&= 2(2 - 1) \\
&= 2
\end{aligned}$$

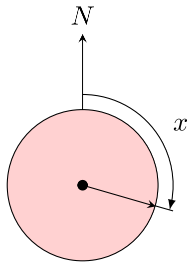
- Compute $V(X)$ using the alternative formula:

$$\begin{aligned}
V(X) &= \int_{1}^{2} x^2 \cdot f(x)\,dx - [E(X)]^2 \\
&= 2 - (2\ln 2)^2 \\
&= 2 - 4\ln^2 2
\end{aligned}$$

A.5 CONTINUOUS UNIFORM DISTRIBUTION

A.5.1 EXPLORING THE CONTINUOUS UNIFORM DISTRIBUTION

Ex 28: Consider a random experiment where a spinner is rotated, and the continuous random variable X represents the angle spun, measured in degrees, over the interval $[0, 360]$.



1. Determine the probability density function of X.
2. Calculate $P(90 \leq X \leq 180)$.
3. Calculate $P(X \geq 60)$.
4. Calculate the expected value $E(X)$.

Answer:

1. Since X represents the angle spun by a fair spinner over $[0, 360]$, it follows a continuous uniform distribution. The probability density function is constant over the interval, with total length $360 - 0 = 360$. Thus, $f(x) = \frac{1}{360}$.
2. $$\begin{aligned} P(90 \leq X \leq 180) &= \int_{90}^{180} f(x)\,\mathrm{d}x \\ &= \int_{90}^{180} \frac{1}{360}\,\mathrm{d}x \\ &= \left[\frac{1}{360}x\right]_{90}^{180} \\ &= \frac{1}{360}(180-90) \\ &= \frac{90}{360} \\ &= \frac{1}{4} \end{aligned}$$
3. $$\begin{aligned} P(X \geq 60) &= \int_{60}^{360} f(x)\,\mathrm{d}x \\ &= \int_{60}^{360} \frac{1}{360}\,\mathrm{d}x \\ &= \left[\frac{1}{360}x\right]_{60}^{360} \\ &= \frac{1}{360}(360-60) \\ &= \frac{300}{360} \\ &= \frac{5}{6} \end{aligned}$$
4. $$\begin{aligned} E(X) &= \frac{0+360}{2} \\ &= 180 \end{aligned}$$

Ex 29: Consider a scenario where the continuous random variable X represents the waiting time at a bus stop, uniformly distributed over the interval $[0, 10]$ minutes.

1. Determine the probability density function of X.
2. Calculate $P(X \leq 8)$.
3. Calculate the expected value $E(X)$.

Answer:

1. Since X represents the waiting time uniformly distributed over $[0, 10]$, it follows a continuous uniform distribution. The total length of the interval is $10 - 0 = 10$. Thus, the probability density function (PDF) is $f(x) = \frac{1}{10}$ for $0 \leq x \leq 10$, and $f(x) = 0$ otherwise.
2. $$\begin{aligned} P(X \leq 8) &= \int_{0}^{8} f(x)\,\mathrm{d}x \\ &= \int_{0}^{8} \frac{1}{10}\,\mathrm{d}x \\ &= \left[\frac{1}{10}x\right]_{0}^{8} \\ &= \frac{1}{10}(8-0) \\ &= \frac{8}{10} \\ &= \frac{4}{5} \end{aligned}$$
3. $$\begin{aligned} E(X) &= \frac{0+10}{2} \\ &= 5 \end{aligned}$$

Ex 30: Let X be a continuous random variable following a continuous uniform distribution on $[a, b]$.
Prove that for all $c, d \in [a, b]$,

$$P(c \leq X \leq d) = \frac{d-c}{b-a}.$$

Answer: Since X is uniformly distributed over $[a, b]$, its probability density function is $f(x) = \frac{1}{b-a}$.
Let $c, d \in [a, b]$.

$$\begin{aligned} P(c \leq X \leq d) &= \int_{c}^{d} f(x)\,\mathrm{d}x \\ &= \int_{c}^{d} \frac{1}{b-a}\,\mathrm{d}x \\ &= \left[\frac{x}{b-a}\right]_{c}^{d} \\ &= \frac{d}{b-a} - \frac{c}{b-a} \\ &= \frac{d-c}{b-a}. \end{aligned}$$

Ex 31: Let X be a continuous random variable following a continuous uniform distribution on $[a, b]$.
Prove that the expected value of X is:

$$E(X) = \frac{a+b}{2}.$$

Answer: Since X is uniformly distributed over $[a, b]$, its probability density function is $f(x) = \frac{1}{b-a}$.

$$\begin{aligned}
E(X) &= \int_a^b x f(x)\,\mathrm{d}x \\
&= \int_a^b x \cdot \frac{1}{b-a}\,\mathrm{d}x \\
&= \frac{1}{b-a}\int_a^b x\,\mathrm{d}x \\
&= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b \\
&= \frac{1}{b-a}\left(\frac{b^2}{2} - \frac{a^2}{2}\right) \\
&= \frac{1}{b-a}\cdot\frac{b^2-a^2}{2} \\
&= \frac{b^2-a^2}{2(b-a)} \\
&= \frac{(b-a)(b+a)}{2(b-a)} \quad \text{(since } b^2-a^2=(b-a)(b+a)\text{)} \\
&= \frac{b+a}{2}.
\end{aligned}$$

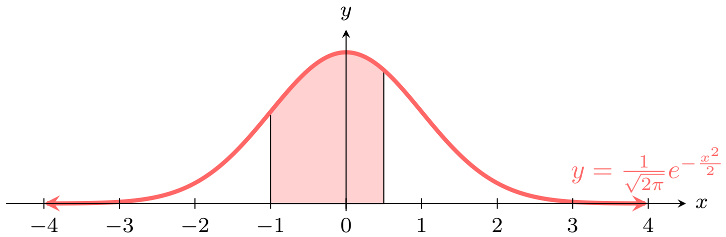
Thus, $E(X) = \frac{a+b}{2}$, which is the midpoint of the interval $[a, b]$.

B NORMAL DISTRIBUTION

B.1 STANDARD NORMAL DISTRIBUTION

B.1.1 FINDING A PROBABILITY FROM AN AREA

MCQ 32: The random variable X follows a standard normal distribution.

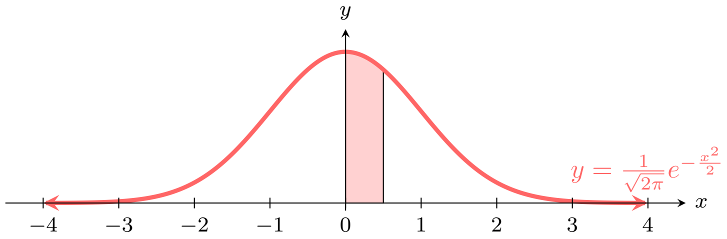


Find the probability corresponding to the red area.
Choose the one correct answer:

- ☐ $P(0 \leq X \leq 0.5)$
- ☒ $P(-1 \leq X \leq 0.5)$
- ☐ $P(X \leq 0.5)$
- ☐ $P(X \geq 1)$
- ☐ $P(X > -0.5)$

Answer: The red area is the integral under the curve over the interval $[-1, \frac{1}{2}]$. By definition, this corresponds to $P(-1 \leq X \leq 0.5)$, making the second option the correct answer.

MCQ 33: The random variable X follows a standard normal distribution.

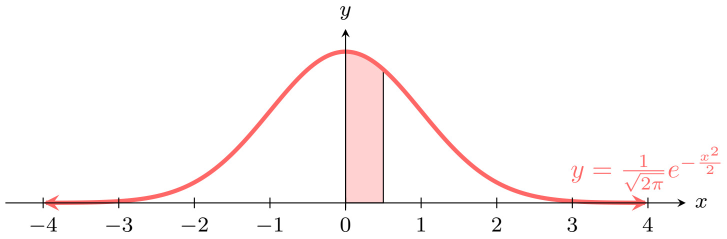


Find the probability corresponding to the red area.
Choose the one correct answer:

- ☒ $P(0 \leq X \leq 0.5)$
- ☐ $P(-1 \leq X \leq 0.5)$
- ☐ $P(X \leq 0.5)$
- ☐ $P(X \geq 1)$
- ☐ $P(X > -0.5)$

Answer: The red area is the integral under the curve over the interval $[0, \frac{1}{2}]$. By definition, this corresponds to $P(0 \leq X \leq 0.5)$, making the first option the correct answer.

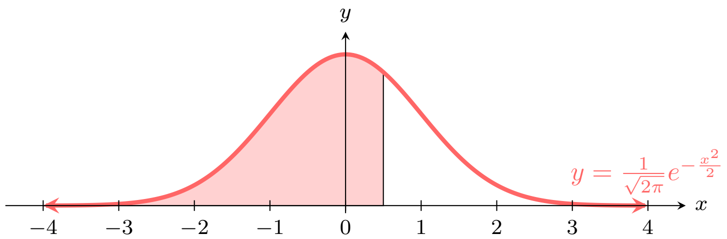
MCQ 34: The random variable X follows a standard normal distribution.



Find the probability corresponding to the red area.
Choose the one correct answer:

- ☒ $P(0 \leqslant X \leqslant 0.5)$
- ☐ $P(-1 \leqslant X \leqslant 0.5)$
- ☐ $P(X \leqslant 0.5)$
- ☐ $P(X \geqslant 1)$
- ☐ $P(X > -0.5)$

MCQ 35: The random variable X follows a standard normal distribution.



Find the probability corresponding to the red area.
Choose the one correct answer:

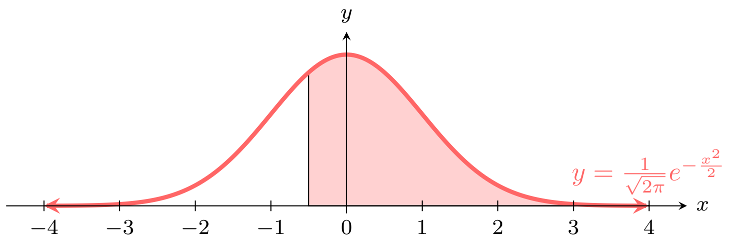
- ☐ $P(0 \leq X \leq 0.5)$
- ☐ $P(-1 \leq X \leq 0.5)$
- ☒ $P(X \leq 0.5)$

☐ $P(X \geq 1)$

☐ $P(X > -0.5)$

Answer: The red area is the integral under the curve from $-\infty$ to $\frac{1}{2}$. By definition, this corresponds to $P(X \leq 0.5)$, making the third option the correct answer.

MCQ 36: The random variable X follows a standard normal distribution.



Find the probability corresponding to the red area.
Choose the one correct answer:

☐ $P(0 \leq X \leq 0.5)$

☐ $P(-1 \leq X \leq 0.5)$

☐ $P(X \leq 0.5)$

☐ $P(X \geq 1)$

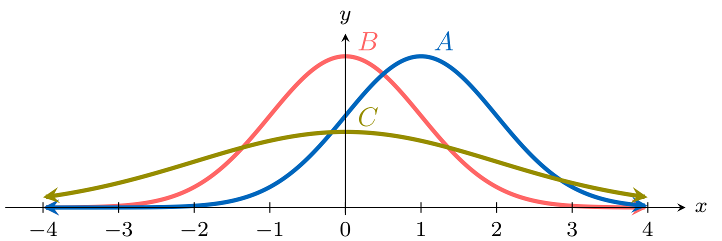
☒ $P(X > -0.5)$

Answer: The red area is the integral under the curve from $-\frac{1}{2}$ to $+\infty$. By definition, this corresponds to $P(X > -0.5)$, making the fifth option the correct answer.

B.2 NORMAL DISTRIBUTION

B.2.1 FINDING THE NORMAL DISTRIBUTION

MCQ 37: Consider three normal distributions A, B, and C, each represented by their probability density functions as shown below.



Identify which normal distribution has a mean of 1 and a standard deviation of 1.
Choose the one correct answer:

☒ Distribution A

☐ Distribution B

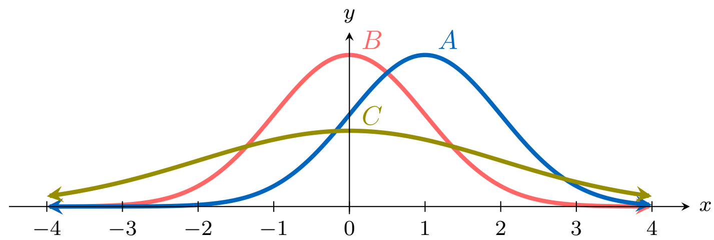
☐ Distribution C

Answer: The normal distribution with a mean of 1 and a standard deviation of 1 is Distribution A. In the diagram:

- Distribution A (blue) is centered at $x = 1$ with a standard deviation of 1, matching $N(1, 1)$.
- Distribution B (red) is centered at $x = 0$ with a standard deviation of 1, representing $N(0, 1)$.
- Distribution C (olive) is centered at $x = 0$ with a standard deviation of 2, representing $N(0, 4)$.

Thus, the correct answer is Distribution A.

MCQ 38: Consider three normal distributions A, B, and C, each represented by their probability density functions as shown below.



Identify which normal distribution has a mean of 0 and a standard deviation of 1.
Choose the one correct answer:

☐ Distribution A

☒ Distribution B

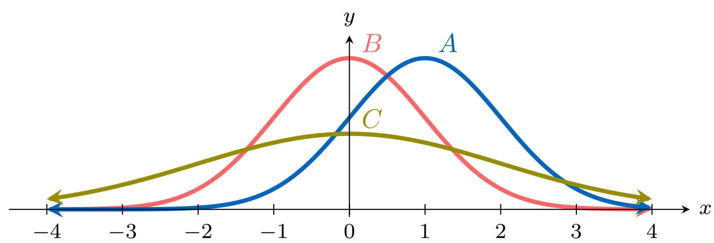
☐ Distribution C

Answer: The normal distribution with a mean of 0 and a standard deviation of 1 is Distribution B. In the diagram:

- Distribution A (blue) is centered at $x = 1$ with a standard deviation of 1, matching $N(1, 1)$.
- Distribution B (red) is centered at $x = 0$ with a standard deviation of 1, representing $N(0, 1)$.
- Distribution C (olive) is centered at $x = 0$ with a standard deviation of 2, representing $N(0, 4)$.

Thus, the correct answer is Distribution B.

MCQ 39: Consider three normal distributions A, B, and C, each represented by their probability density functions as shown below.



Identify which normal distribution has a mean of 0 and a standard deviation of 2.
Choose the one correct answer:

☐ Distribution A

☐ Distribution B

☒ Distribution C

Answer: The normal distribution with a mean of 0 and a standard deviation of 2 is Distribution C. In the diagram:

- Distribution A (blue) is centered at $x = 1$ with a standard deviation of 1, matching $N(1, 1)$.
- Distribution B (red) is centered at $x = 0$ with a standard deviation of 1, representing $N(0, 1)$.
- Distribution C (olive) is centered at $x = 0$ with a standard deviation of 2, representing $N(0, 4)$.

Thus, the correct answer is Distribution C.

B.2.2 FINDING VALUES USING THE MEAN AND STANDARD DEVIATION

Ex 40: The height of one-year-old babies is normally distributed with a mean of 75 cm and a standard deviation of 3 cm. For medical purposes, a doctor needs to determine the height that corresponds to one standard deviation above the mean.

$$\boxed{78}\text{ cm}$$

Answer: Given a normal distribution with a mean (μ) of 75 cm and a standard deviation (σ) of 3 cm, one standard deviation above the mean is calculated as:

$$\mu + \sigma = 75 + 3 = 78\,\text{cm}.$$

Thus, the height is 78 cm.

Ex 41: In a gas at thermal equilibrium, the velocities of particles follow a normal distribution with a mean velocity of 500 m/s and a standard deviation of 100 m/s. A physicist wants to calculate the velocity that corresponds to one standard deviation below the mean.

$$\boxed{400}\text{ m/s}$$

Answer: Given a normal distribution with a mean (μ) of 500 m/s and a standard deviation (σ) of 100 m/s, one standard deviation below the mean is calculated as:

$$\mu - \sigma = 500 - 100 = 400\,\text{m/s}.$$

Thus, the velocity is 400 m/s.

Ex 42: The weight of adult women is normally distributed with a mean of 65 kg and a standard deviation of 5 kg. For a health study, a researcher needs to determine the weight that corresponds to two standard deviations above the mean.

$$\boxed{75}\text{ kg}$$

Answer: Given a normal distribution with a mean (μ) of 65 kg and a standard deviation (σ) of 5 kg, two standard deviations above the mean is calculated as:

$$\mu + 2\sigma = 65 + 2 \cdot 5 = 65 + 10 = 75\,\text{kg}.$$

Thus, the weight is 75 kg.

Ex 43: The final exam scores in a math course are normally distributed with a mean of 70 points and a standard deviation of 8 points. A teacher wants to identify students who scored one standard deviation below the mean.

$$\boxed{62}\text{ points}$$

Answer: Given a normal distribution with a mean (μ) of 70 points and a standard deviation (σ) of 8 points, one standard deviation below the mean is calculated as:

$$\mu - \sigma = 70 - 8 = 62\,\text{points}.$$

Thus, the score is 62 points.

B.2.3 FINDING PROBABILITIES USING GRAPHIC CALCULATOR

Ex 44: Suppose X represents the time (in minutes) taken to complete a task, and it follows a normal distribution with a mean of 40 minutes and a standard deviation of 10 minutes. Calculate the probability that the task is completed between 37 and 48 minutes. Round your answer to three decimal places.

$$P(37 \leqslant X \leqslant 48) \approx \boxed{0.406}$$

Answer: Given $X \sim \mathcal{N}(40, 10^2)$, where $\mu = 40$ (mean) and $\sigma = 10$ (standard deviation), we calculate $P(37 \leqslant X \leqslant 48)$ using a graphing calculator. Set the lower limit to 37, the upper limit to 48, $\mu = 40$, and $\sigma = 10$. The calculator computes the area under the normal curve between these bounds, yielding:

$$P(37 \leqslant X \leqslant 48) \approx 0.406$$

This result indicates that there is approximately a 40.6% chance that the task completion time falls between 37 and 48 minutes.

Ex 45: Suppose X represents the annual rainfall (in millimeters) in a coastal city, and it follows a normal distribution with a mean of 1200 mm and a standard deviation of 150 mm. Calculate the probability that the annual rainfall exceeds 1350 mm. Round your answer to two decimal places.

$$P(X \geqslant 1350) \approx \boxed{0.16}$$

Answer: Given $X \sim \mathcal{N}(1200, 150^2)$, where $\mu = 1200$ (mean) and $\sigma = 150$ (standard deviation), we calculate $P(X \geqslant 1350)$ using a graphing calculator. Set the lower limit to 1350, the upper limit to infinity (or a very large number, e.g., 10^6), $\mu = 1200$, and $\sigma = 150$. The calculator computes the area under the normal curve to the right of 1350, yielding:

$$P(X \geqslant 1350) \approx 0.16$$

This result indicates that there is approximately a 16% chance that the annual rainfall exceeds 1350 mm in this city.

Ex 46: Suppose X represents the Elo rating of a chess player, and it follows a normal distribution with a mean of 1500 and a standard deviation of 200. Calculate the probability that a player's rating exceeds 2000. Round your answer to three decimal places.

$$P(X \geqslant 2000) \approx \boxed{0.006}$$

Answer: Given $X \sim \mathcal{N}(1500, 200^2)$, where $\mu = 1500$ (mean) and $\sigma = 200$ (standard deviation), we calculate $P(X \geqslant 2000)$ using a graphing calculator. Set the lower limit to 2000, the upper limit to infinity (or a very large number, e.g., 10^6), $\mu = 1500$, and $\sigma = 200$. The calculator computes the area under the normal curve to the right of 2000, yielding:

$$P(X \geqslant 2000) \approx 0.006$$

This result indicates that there is approximately a 0.6% chance that a player has an Elo rating exceeding 2000.

Ex 47: Suppose X represents the height (in centimeters) of adult women in Australia, and it follows a normal distribution with a mean of 165 cm and a standard deviation of 7 cm. Calculate the probability that a woman's height is less than or equal to 160 cm. Round your answer to three decimal places.

$$P(X \leqslant 160) \approx \boxed{0.238}$$

Answer: Given $X \sim \mathcal{N}(165, 7^2)$, where $\mu = 165$ (mean) and $\sigma = 7$ (standard deviation), we calculate $P(X \leqslant 160)$ using a graphing calculator. Set the lower limit to negative infinity (or a very small number, e.g., -10^6), the upper limit to 160, $\mu = 165$, and $\sigma = 7$. The calculator computes the area under the normal curve to the left of 160, yielding:

$$P(X \leqslant 160) \approx 0.238$$

This result indicates that there is approximately a 23.8% chance that an adult woman in Australia has a height of 160 cm or less.

B.2.4 BUSTING BRAGS AND CLAIMS WITH NORMAL CURVES

Ex 48: Suppose X represents the scores (in points) of students in a math class evaluation, and it follows a normal distribution with a mean of 65 points and a standard deviation of 10 points. Hugo receives a score of 75 points and claims, "I am in the top 2% of students in this class."
Do you agree with Hugo? Explain your answer.

Answer: Given $X \sim \mathcal{N}(65, 10^2)$, where $\mu = 65$ (mean) and $\sigma = 10$ (standard deviation), we calculate $P(X > 75)$ to check Hugo's claim of being in the top 2%. Using a graphing calculator, set the lower limit to 75, $\mu = 65$, and $\sigma = 10$. The calculator gives:

$$P(X \geq 75) \approx 0.16$$

This result means that approximately 16% of students score above 75 points. Hugo claims to be in the top 2%. Since 16% is much greater than 2%, I do not agree with Hugo. His score places him in the top 16% of the class, not the top 2%.

Ex 49: Suppose X represents the daily water consumption (in liters) of households in a small town, and it follows a normal distribution with a mean of 200 liters and a standard deviation of 30 liters. Maria measures her household's consumption as 260 liters and claims, "We are in the top 2% of households in this town."
Do you agree with Maria's claim? Explain your answer.

Answer: Given $X \sim \mathcal{N}(200, 30^2)$, where $\mu = 200$ (mean) and $\sigma = 30$ (standard deviation), we calculate $P(X \geq 260)$ to check Maria's claim of being in the top 2%. Using a graphing calculator, set the lower limit to 260, $\mu = 200$, and $\sigma = 30$. The calculator gives:

$$P(X \geq 260) \approx 0.023$$

I do agree. She's in the top 2.3%, almost 2%.

Ex 50: Suppose X represents the height (in centimeters) of boys in a school, and it follows a normal distribution with a mean of 175 cm and a standard deviation of 8 cm. The school states, "95% of boys can pass under a door of height 190 cm."
Do you agree with this statement? Explain your answer.

Answer: Given $X \sim \mathcal{N}(175, 8^2)$, where $\mu = 175$ (mean) and $\sigma = 8$ (standard deviation), we calculate $P(X \leq 190)$ to check the claim that 95% of boys are under 190 cm. Using a graphing calculator, set the lower limit to negative infinity (or a very small number, e.g., -10^6), the upper limit to 190, $\mu = 175$, and $\sigma = 8$. The calculator gives:

$$P(X \leq 190) \approx 0.969$$

This result means approximately 96.9% of boys have a height of 190 cm or less. The claim states 95%. Since 96.9% is close to 95%, I agree with the statement, though it slightly underestimates the true percentage.

Ex 51: Suppose X represents the high scores (in points) of players in a new battle royale video game, and it follows a normal distribution with a mean of 500 points and a standard deviation of 50 points. Liam gets a high score of 600 points and brags, "I'm in the top 5% of all players!"
Do you agree with Liam? Explain your answer.

Answer: Given $X \sim \mathcal{N}(500, 50^2)$, where $\mu = 500$ (mean) and $\sigma = 50$ (standard deviation), we calculate $P(X \geq 600)$ to check Liam's claim of being in the top 5%. Using a graphing calculator, set the lower limit to 600, the upper limit to infinity (or a very large number, e.g., 10^6), $\mu = 500$, and $\sigma = 50$. The calculator gives:

$$P(X \geq 600) \approx 0.023$$

This result means approximately 2.3% of players score 600 points or higher. Liam claims he's in the top 5% (i.e., $P(X \geq 600) \leq 0.05$). Since 2.3% is less than 5%, I don't agree with Liam's exact claim—he's actually in the top 2.3%, which is even better than he thinks!

B.3 EMPIRICAL RULE FOR NORMAL DISTRIBUTION

B.3.1 EXPLORING EVERYDAY STATISTICS

Ex 52: Height and weight are key measurements for tracking a child's development. The World Health Organization assesses child development by comparing the weights of children of the same height and gender. In 2009, the weights of all 80 cm girls in a reference population were normally distributed with a mean of 10.2 kg and a standard deviation of 0.8 kg.
Using this information, calculate the following probabilities or values for the weights of 80 cm girls:

1. The percentage of girls with weights between 10.2 kg and 11 kg.

   $\boxed{34.13}$ %

2. The percentage of girls with weights between 10.2 kg and 11.8 kg.

   $\boxed{47.72}$ %

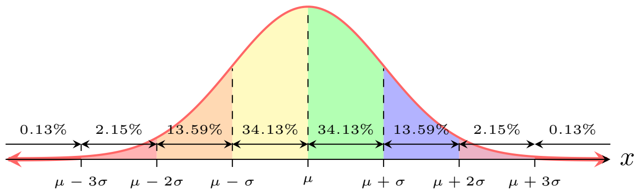
3. The percentage of girls with weights greater than 9.4 kg.

   $\boxed{84.13}$ %

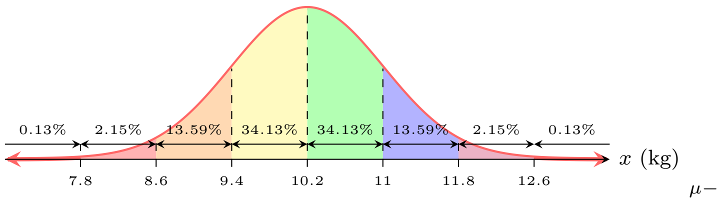
4. In 2010, if there were 545 girls who were 80 cm tall, estimate the number of girls with weights between 9.4 kg and 11 kg (round to the nearest integer).

   $\boxed{372}$ girls

For a normal distribution, the coverage probabilities are illustrated below:



Answer:



3σ=7.8, $\mu-2\sigma$=8.6, $\mu-\sigma$=9.4, μ=10.2, $\mu+\sigma$=11, $\mu+2\sigma$=11.8

1. Approximately 34.13% of girls have weights between 10.2 kg and 11 kg (from μ to $\mu+\sigma$).
2. Approximately 47.72% of girls have weights between 10.2 kg and 11.8 kg (from μ to $\mu+2\sigma$, i.e., 34.13% + 13.59%).
3. Approximately 84.13% of girls have weights greater than 9.4 kg (from $\mu-\sigma$ to infinity, i.e., 34.13% + 50%).
4. For weights between 9.4 kg and 11 kg (from $\mu-\sigma$ to $\mu+\sigma$), the percentage is 68.26% (34.13% + 34.13%). Thus, for 545 girls: $545 \times 0.6826 \approx 372$.

Ex 53: Exam scores are a key measure for evaluating student performance. A national education board assesses student achievement by analyzing scores from a standardized test. In 2023, the scores of all students in a particular grade were normally distributed with a mean of 75 points and a standard deviation of 5 points.
Using this information, calculate the following probabilities or values for the students' scores:

1. The percentage of students with scores between 70 and 75 points.

   [34.13] %

2. The percentage of students with scores between 65 and 75 points.

   [47.72] %

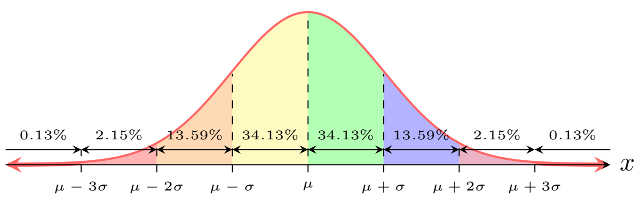
3. The percentage of students with scores less than 80 points.

   [84.13] %

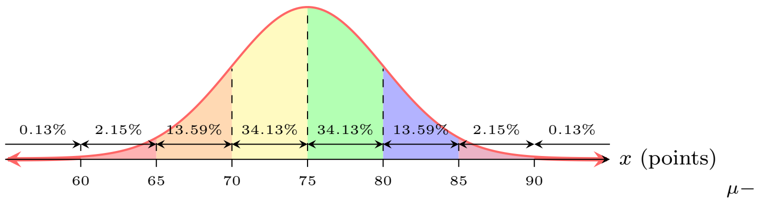
4. In 2024, if there were 600 students in this grade, estimate the number of students with scores between 70 and 85 points (round to the nearest integer).

   [491] students

For a normal distribution, the coverage probabilities are illustrated below:



Answer:



3σ=60, $\mu-2\sigma$=65, $\mu-\sigma$=70, μ=75, $\mu+\sigma$=80, $\mu+2\sigma$=85

1. Approximately 34.13% of students have scores between 70 and 75 points (from $\mu-\sigma$ to μ).
2. Approximately 47.72% of students have scores between 65 and 75 points (from $\mu-2\sigma$ to μ, i.e., 13.59% + 34.13%).
3. Approximately 84.13% of students have scores less than 80 points (from $-\infty$ to $\mu+\sigma$, i.e., 50% + 34.13%).
4. For scores between 70 and 85 points (from $\mu-\sigma$ to $\mu+2\sigma$), the percentage is 81.85% (34.13% + 34.13% + 13.59%). Thus, for 600 students: $600 \times 0.8185 \approx 491$.

Ex 54: Intelligence Quotient (IQ) scores are widely used to measure cognitive ability. A psychological research institute analyzes IQ scores to understand population intelligence distributions. In 2023, the IQ scores of a large adult population were normally distributed with a mean of 100 and a standard deviation of 15.
Using this information, calculate the following probabilities or values for the IQ scores:

1. The percentage of adults with IQ scores between 85 and 100.

   [34.13] %

2. The percentage of adults with IQ scores between 70 and 100.

   [47.72] %

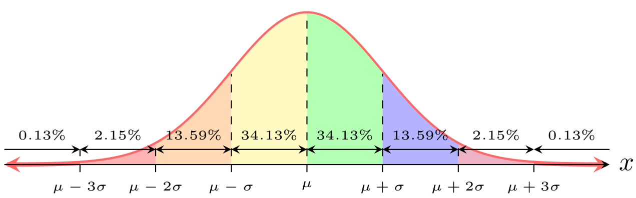
3. The percentage of adults with IQ scores less than 115.

   [84.13] %

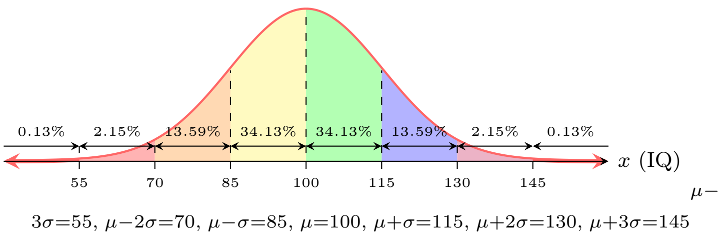
4. In 2024, if there were 800 adults in this population, estimate the number of adults with IQ scores greater than 130 (round to the nearest integer).

   [18] adults

For a normal distribution, the coverage probabilities are illustrated below:



Answer:



1. Approximately 34.13% of adults have IQ scores between 85 and 100 (from $\mu - \sigma$ to μ).
2. Approximately 47.72% of adults have IQ scores between 70 and 100 (from $\mu - 2\sigma$ to μ, i.e., 13.59% + 34.13%).
3. Approximately 84.13% of adults have IQ scores less than 115 (from $-\infty$ to $\mu + \sigma$, i.e., 50% + 34.13%).
4. For IQ scores greater than 130 (above $\mu+2\sigma$), the percentage is 2.28% (100% - 97.72%, where 97.72% = 50% + 34.13% + 13.59%). Thus, for 800 adults: $800 \times 0.0228 \approx 18.24$, rounded to 18.

Ex 55: Daily screen time is a critical metric for understanding teenage behavior and well-being. A national health study investigates the amount of time teenagers spend on screens (e.g., phones, computers, TVs) per day. In 2023, the daily screen time of teenagers in a large sample was normally distributed with a mean of 6 hours and a standard deviation of 1.5 hours.
Using this information, calculate the following probabilities or values for the daily screen time of teenagers:

1. The percentage of teenagers with daily screen time between 4.5 and 6 hours.

   34.13 %

2. The percentage of teenagers with daily screen time between 6 and 9 hours.

   47.72 %

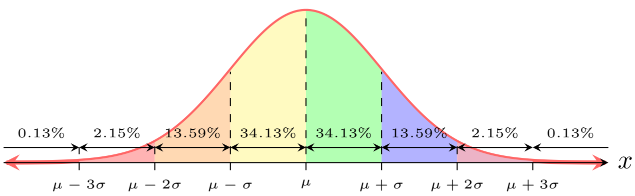
3. The percentage of teenagers with daily screen time less than 7.5 hours.

   84.13 %

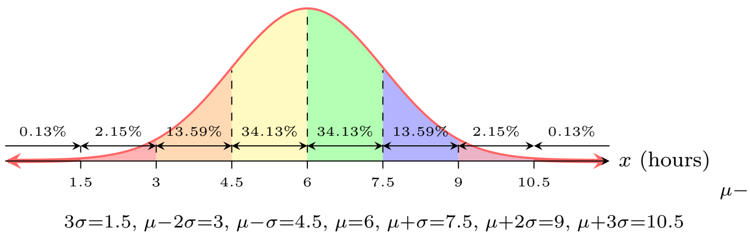
4. In 2024, if there were 1200 teenagers in this sample, estimate the number of teenagers with daily screen time greater than 9 hours (round to the nearest integer).

   27 teenagers

For a normal distribution, the coverage probabilities are illustrated below:



Answer:



1. Approximately 34.13% of teenagers have daily screen time between 4.5 and 6 hours (from $\mu - \sigma$ to μ).
2. Approximately 47.72% of teenagers have daily screen time between 6 and 9 hours (from μ to $\mu + 2\sigma$, i.e., 34.13% + 13.59%).
3. Approximately 84.13% of teenagers have daily screen time less than 7.5 hours (from $-\infty$ to $\mu + \sigma$, i.e., 50% + 34.13%).
4. For daily screen time greater than 9 hours (above $\mu + 2\sigma$), the percentage is 2.28% (100% - 97.72%, where 97.72% = 50% + 34.13% + 13.59%). Thus, for 1200 teenagers: $1200 \times 0.0228 \approx 27.36$, rounded to 27.

B.4 QUANTILES

B.4.1 SETTING THE THRESHOLD WITH PERCENTILES

Ex 56: Suppose X represents the time (in minutes) taken to complete a task, and it follows a normal distribution with a mean of 40 minutes and a standard deviation of 10 minutes. The teacher fixes the duration of the exam such that 95% of students have finished. Find this time (i.e., the 95th percentile). Round your answer to one decimal place.

$x \approx$ 56.4

Answer: Given $X \sim \mathcal{N}(40, 10^2)$, where $\mu = 40$ (mean) and $\sigma = 10$ (standard deviation), we need to find the time x such that $P(X \leq x) = 0.95$, the 95th percentile, which is the exam duration set by the teacher for 95% of students to finish. Using a graphing calculator's inverse normal function (e.g., 'invNorm'), input the probability 0.95, $\mu = 40$, and $\sigma = 10$. The calculator yields:

$$x \approx 56.4$$

This result means that if the teacher sets the exam duration to 56.4 minutes, 95% of students will have finished the task.

Ex 57: Suppose X represents the delivery time (in minutes) of pizzas from a local shop, and it follows a normal distribution with a mean of 25 minutes and a standard deviation of 5 minutes. The shop guarantees a delivery deadline such that 90% of orders are delivered before this time. Find this time (i.e., the 90th percentile). Round your answer to one decimal place.

$x \approx$ 31.4

Answer: Given $X \sim \mathcal{N}(25, 5^2)$, where $\mu = 25$ (mean) and $\sigma = 5$ (standard deviation), we need to find the time x such that $P(X \leq x) = 0.90$, the 90th percentile, which is the delivery deadline for 90% of orders to be completed. Using a graphing calculator's inverse normal function (e.g., ‘invNorm‘), input the probability 0.90, $\mu = 25$, and $\sigma = 5$. The calculator yields:

$$x \approx 31.4$$

This result means that if the shop sets the delivery deadline to 31.4 minutes, 90% of pizza orders will be delivered before this time.

Ex 58: Suppose X represents the height (in centimeters) of men, and it follows a normal distribution with a mean of 175.3 cm and a standard deviation of 7.1 cm. A builder wants to design a door height such that at least 95% of men can walk through without ducking. Find this height (i.e., the 95th percentile). Round your answer to one decimal place.

$$x \approx \boxed{187.0}$$

Answer: Given $X \sim \mathcal{N}(175.3, 7.1^2)$, where $\mu = 175.3$ (mean) and $\sigma = 7.1$ (standard deviation), we need to find the height x such that $P(X \leq x) = 0.95$, the 95th percentile, which is the door height allowing at least 95% of men to walk through without ducking. Using a graphing calculator's inverse normal function (e.g., ‘invNorm‘), input the probability 0.95, $\mu = 175.3$, and $\sigma = 7.1$. The calculator yields:

$$x \approx 187.0$$

This result means that if the builder sets the door height to 187.0 cm, at least 95% of men will be able to walk through without ducking.

Ex 59: Suppose X represents the battery life (in hours) of a new smartphone model, and it follows a normal distribution with a mean of 12 hours and a standard deviation of 2 hours. The manufacturer sets a warranty replacement time such that 80% of phones last at least this long before needing a recharge. Find this time (i.e., the 20th percentile, since it's the lower tail). Round your answer to one decimal place.

$$x \approx \boxed{10.3}$$

Answer: Given $X \sim \mathcal{N}(12, 2^2)$, where $\mu = 12$ (mean) and $\sigma = 2$ (standard deviation), we need to find the time x such that $P(X \geq x) = 0.80$, which means $P(X \leq x) = 0.20$, the 20th percentile, since it's the lower bound for 80% lasting longer. Using a graphing calculator's inverse normal function (e.g., ‘invNorm‘), input the probability 0.20, $\mu = 12$, and $\sigma = 2$. The calculator yields:

$$x \approx 10.3$$

This result means that if the manufacturer sets the warranty time to 10.3 hours, 80% of phones will last at least this long before needing a recharge.

Ex 60: Suppose X represents the noise level (in decibels) of a crowd at a school concert, and it follows a normal distribution with a mean of 85 decibels and a standard deviation of 15 decibels. The sound engineer sets a microphone threshold such that 60% of the time, the noise is below this level. Find this noise level (i.e., the 60th percentile). Round your answer to one decimal place.

$$x \approx \boxed{88.8}$$

Answer: Given $X \sim \mathcal{N}(85, 15^2)$, where $\mu = 85$ (mean) and $\sigma = 15$ (standard deviation), we need to find the noise level x such that $P(X \leq x) = 0.60$, the 60th percentile, which is the threshold for 60% of the noise levels to be below it. Using a graphing calculator's inverse normal function (e.g., ‘invNorm‘), input the probability 0.60, $\mu = 85$, and $\sigma = 15$. The calculator yields:

$$x \approx 88.8$$

This result means that if the sound engineer sets the microphone threshold to 88.8 decibels, 60% of the time the crowd noise will be below this level.

Ex 61: Suppose X represents the weight (in kilograms) of backpacks carried by students, and it follows a normal distribution with a mean of 8 kg and a standard deviation of 1.5 kg. The school sets a minimum weight limit for a strength training program such that 95% of students carry at least this weight. Find this weight (i.e., the 5th percentile, since it's the lower tail). Round your answer to one decimal place.

$$x \approx \boxed{5.5}$$

Answer: Given $X \sim \mathcal{N}(8, 1.5^2)$, where $\mu = 8$ (mean) and $\sigma = 1.5$ (standard deviation), we need to find the weight x such that $P(X \geq x) = 0.95$, which means $P(X \leq x) = 0.05$, the 5th percentile, since it's the lower bound for 95% carrying at least this weight. Using a graphing calculator's inverse normal function (e.g., ‘invNorm‘), input the probability 0.05, $\mu = 8$, and $\sigma = 1.5$. The calculator yields:

$$x \approx 5.5$$

This result means that if the school sets the minimum weight limit to 5.5 kg, 95% of students will carry backpacks weighing at least this much.