COMPLEX NUMBERS: GEOMETRICAL APPROACH

While the algebraic approach provides the rules for manipulating complex numbers, the geometrical approach offers a powerful and intuitive way to understand their meaning. By representing complex numbers as points or vectors in a plane, we can visualise their operations as geometric transformations such as rotations, reflections, and scalings. This chapter explores the geometry of the complex plane, providing insight into the concepts of modulus, argument, and the geometric patterns formed by the roots of complex numbers.

A COMPLEX PLANE

We have seen that a complex number can be written in Cartesian (algebraic) form as $z = a + bi$ where $a = \text{Re}(z)$ and $b = \text{Im}(z)$ are both real numbers.

There is a one-to-one correspondence between any complex number $a + bi$ and the point (a, b).

We can therefore plot any complex number on a plane by associating it with a unique point, giving a geometric representation to what was previously an algebraic object. We refer to this plane as the **complex plane**.
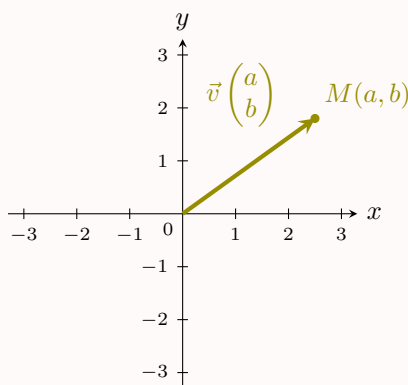
> Definition **Complex Plane**
>
> The **complex plane** is the plane formed by the complex numbers, endowed with a Cartesian coordinate system such that:
>
> - the horizontal x-axis, called the **real axis**, consists of the real numbers
>
> - the vertical y-axis, called the **imaginary axis**, consists of the purely imaginary numbers.
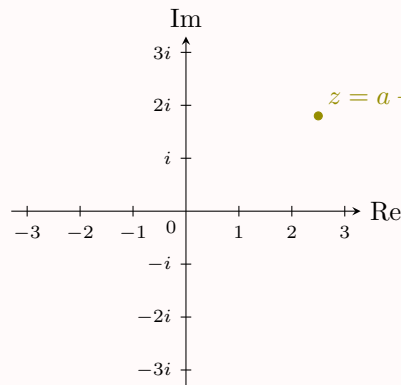>
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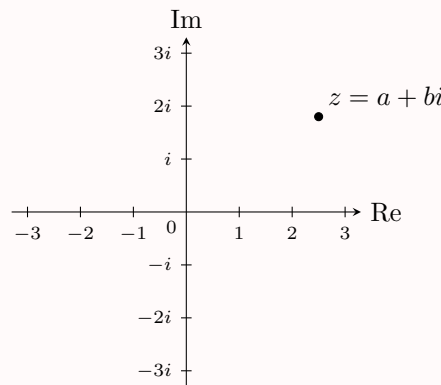
> Definition **Relationship between Cartesian and Complex Representations**
>
> - The point $M(a, b)$ is the **point corresponding to** the complex number $z = a + bi$.
>
> - The vector $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ is the **position vector representing** the complex number $z = a + bi$.
>
> - The complex number $z = a + bi$ is the **affix** of the point $M(a, b)$ and of the vector $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$.
>
>
>
> Cartesian Plane   Complex Plane

B MODULUS AND ARGUMENT

Geometrically, every complex number has two key characteristics: its distance from the origin and its direction relative to the positive real axis.

The **modulus** of a complex number $z = x + yi$, denoted $|z|$, is the distance from the origin to its corresponding point in the complex plane. By the Pythagorean theorem, it is defined as:
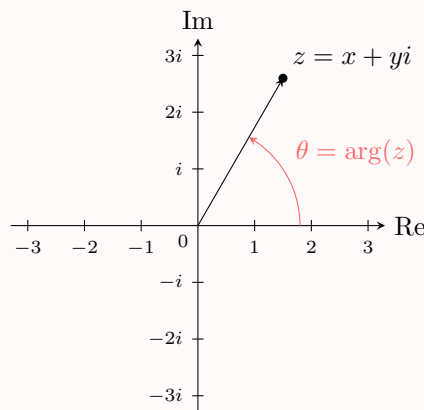
$$|z| = \sqrt{x^2 + y^2}.$$

The modulus is a non-negative real number.
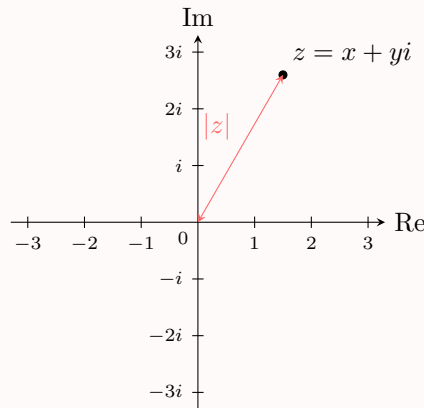
Definition **Argument**

The **argument** of a non-zero complex number z, denoted $\arg(z)$, is any real number θ such that θ is the angle measured anti-clockwise between the positive real axis and the vector representing z.

Note An argument of a complex number is not unique, as any multiple of 2π can be added to the angle. To make it unique, we define the **principal value** of the argument as the value of θ in the interval $(-\pi, \pi]$.
The argument of 0 is *undefined*.

Ex: Non-zero purely imaginary numbers have an argument of $\dfrac{\pi}{2}$ or $-\dfrac{\pi}{2}$.
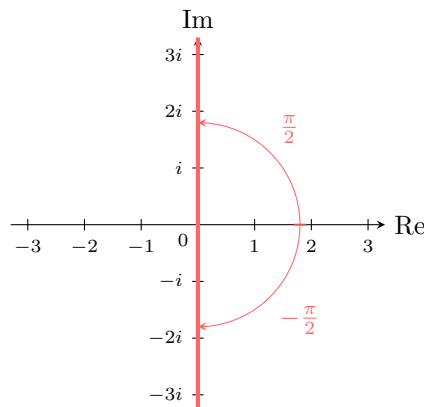
Method **Calculating the Argument**

To find an argument θ of a non-zero complex number $z = a + bi$:

1. **Calculate the modulus** $|z| = \sqrt{a^2 + b^2}$.

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2. **Factor out the modulus** to identify the coordinates of the corresponding point on the unit circle:

$$z = |z| \left(\frac{a}{|z|} + i \frac{b}{|z|} \right).$$

3. **Identify the angle.** Find an angle θ that satisfies the system of equations:

$$\begin{cases} \cos(\theta) = \dfrac{a}{|z|} \\ \sin(\theta) = \dfrac{b}{|z|} \end{cases}$$

This value of θ is an argument of z.

Ex: Find an argument of the complex number $z = -1 + i\sqrt{3}$.

Answer:

1. We first calculate the modulus of z:

$$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2.$$
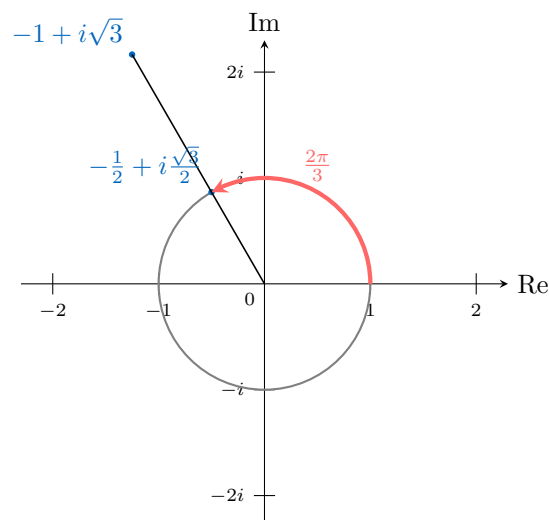
We can then factor out the modulus:

$$z = 2 \times \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2} \right).$$

2. We are looking for a real number θ such that:

$$\begin{cases} \cos(\theta) = -\dfrac{1}{2} \\ \sin(\theta) = \dfrac{\sqrt{3}}{2} \end{cases}$$

The angle $\theta = \dfrac{2\pi}{3}$ satisfies both equations.

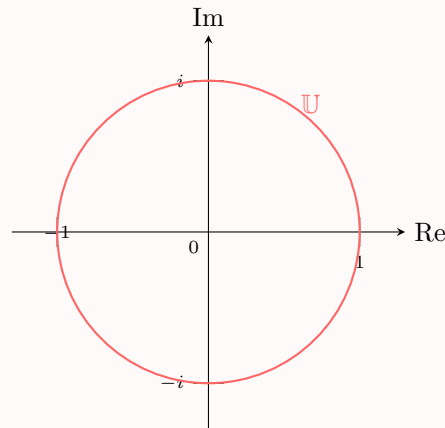Therefore, an argument of z is $\dfrac{2\pi}{3}$.

C UNIT MODULUS COMPLEX NUMBERS AND THE IMAGINARY EXPONENTIAL

> **Definition Unit Circle in the Complex Plane**
>
> The set of all complex numbers with modulus 1 is called the **unit circle**, denoted $\mathbb{U}$:
>
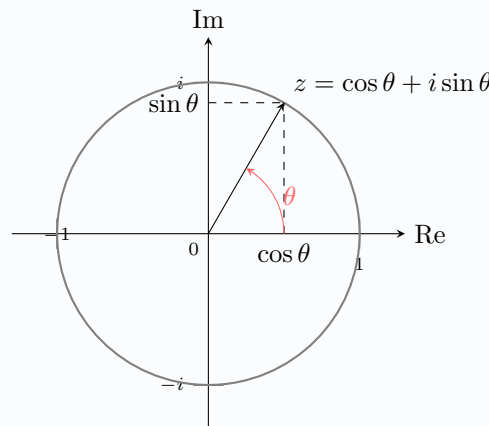> $$\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}.$$
>
>

> **Proposition Trigonometric Form of Unit Modulus Numbers**
>
> If $z \in \mathbb{U}$, then there exists a real number θ such that
>
> $$z = \cos\theta + i\sin\theta.$$
>
> Here, θ is an argument of z.
>
>

Motivation for Euler's Notation Let f be the function defined on $\mathbb{R}$ by

$$f(\theta) = \cos(\theta) + i\sin(\theta).$$

Using the angle addition formulas, one can show that, for all real numbers θ_1 and θ_2,

$$f(\theta_1 + \theta_2) = f(\theta_1)\,f(\theta_2).$$

This is the same fundamental property as the real exponential function ($e^{x+y} = e^x e^y$). This strong analogy motivates the notation

$$f(\theta) = e^{i\theta}.$$

> **Definition Imaginary Exponential**
>
> For any real number θ, we define the **imaginary exponential**, denoted $e^{i\theta}$, as the complex number
>
> $$e^{i\theta} = \cos\theta + i\sin\theta.$$

> **Proposition Properties of the Imaginary Exponential**
>
> For any real numbers θ and θ', and for any integer n, the imaginary exponential has the following properties:

- **Modulus:** $|e^{i\theta}| = 1$

- **Argument:** $\arg(e^{i\theta}) \equiv \theta \pmod{2\pi}$

- **Conjugate:** $\overline{e^{i\theta}} = e^{-i\theta}$

- **Product Rule:** $e^{i(\theta+\theta')} = e^{i\theta}\,e^{i\theta'}$

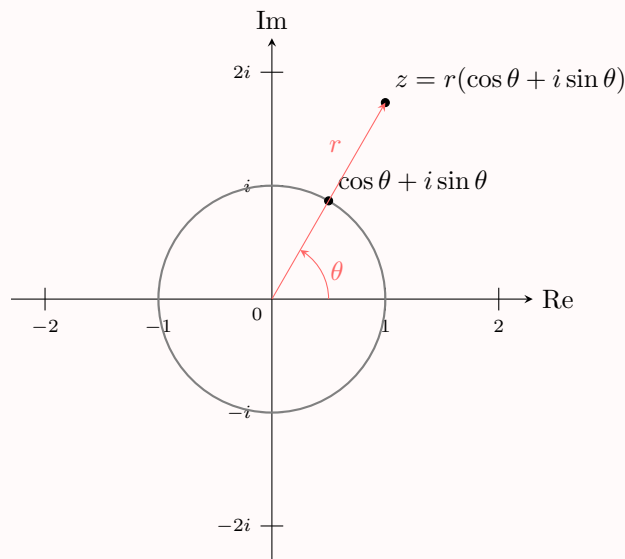- **Power Rule:** $(e^{i\theta})^n = e^{in\theta}$

D POLAR AND EULER'S FORMS

Any non-zero complex number can be represented by starting with a number on the unit circle and scaling it by its modulus.

Definition **Polar Form**

A non-zero complex number z with modulus $r = |z|$ and an argument $\theta \in \arg(z)$ can be written in **polar form** as
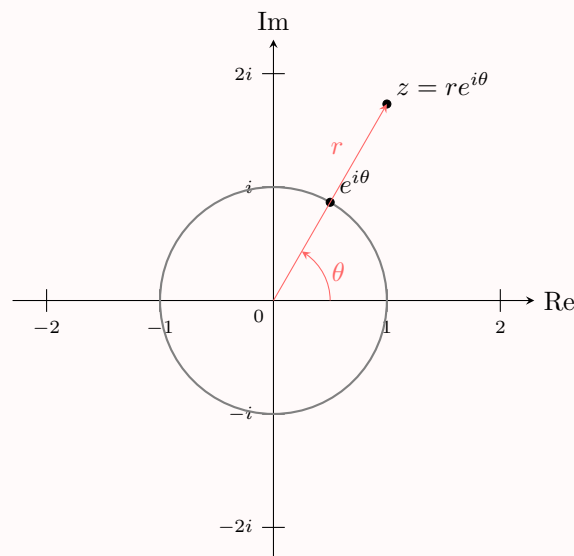
$$z = r(\cos\theta + i\sin\theta).$$

Definition **Euler's Form**

Using the imaginary exponential, a non-zero complex number z with modulus $r = |z|$ and an argument $\theta \in \arg(z)$ can be written in **Euler's form** as

$$z = re^{i\theta}.$$

Note This form makes the geometric rules of multiplication intuitive:

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = (r_1 r_2)e^{i(\theta_1+\theta_2)}.$$

Multiplying complex numbers corresponds to multiplying their moduli and adding their arguments.

E DE MOIVRE'S THEOREM

Proposition De Moivre's Theorem

For any complex number $z = r(\cos\theta + i\sin\theta)$ and for any integer n:

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

In Euler's form, this is expressed concisely as:

$$(re^{i\theta})^n = r^n e^{in\theta}.$$

Proof

The most direct proof uses Euler's form, which extends the familiar laws of exponents to complex numbers. Let $z = re^{i\theta}$. Raising this to the power n gives

$$
\begin{aligned}
z^n &= (re^{i\theta})^n \\
&= r^n \left(e^{i\theta}\right)^n && \text{(power of a product rule)} \\
&= r^n e^{i(n\theta)} && \text{(power of a power rule).}
\end{aligned}
$$

Converting this result back to polar (trigonometric) form gives:

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

This proves the theorem for any integer n.

Ex: Calculating a Power using Euler's Form Calculate $(1 + \sqrt{3}i)^5$.

- **Step 1: Convert to Euler's form.**

$$
\begin{aligned}
r &= |1 + \sqrt{3}i| & \text{and} \quad \theta &= \arg(1 + \sqrt{3}i) \\
&= \sqrt{1^2 + (\sqrt{3})^2} & &= \arctan\left(\frac{\sqrt{3}}{1}\right) \\
&= \sqrt{4} & & \\
&= 2 & &= \frac{\pi}{3}.
\end{aligned}
$$

So, $1 + \sqrt{3}i = 2e^{i\frac{\pi}{3}}$.

- **Step 2: Apply De Moivre's Theorem and convert back.**

Using De Moivre's theorem and the identity $e^{i\varphi} = \cos\varphi + i\sin\varphi$:

$$
\begin{aligned}
(1 + \sqrt{3}i)^5 &= \left(2e^{i\frac{\pi}{3}}\right)^5 \\
&= 2^5 e^{i\frac{5\pi}{3}} \\
&= 32\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right) \\
&= 32\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \\
&= 16 - 16\sqrt{3}i.
\end{aligned}
$$

F PROPERTIES OF MODULUS AND ARGUMENT

Proposition Properties of the Modulus

- $|\bar{z}| = |z|$

- $|z|^2 = z\bar{z}$

- $|z_1 z_2| = |z_1|\,|z_2|$

- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ provided $z_2 \neq 0$

- $|z^n| = |z|^n$ for any integer n.

These properties can be rigorously proven using the algebraic definition of the modulus. The proofs are left as exercises.

Proposition Properties of Arguments

For any non-zero complex numbers z_1 and z_2 and any integer n:

- for any non-zero complex number z, $\arg(\overline{z}) \equiv -\arg(z) \pmod{2\pi}$

- $\arg(z_1 z_2) \equiv \arg(z_1) + \arg(z_2) \pmod{2\pi}$

- $\arg\left(\dfrac{z_1}{z_2}\right) \equiv \arg(z_1) - \arg(z_2) \pmod{2\pi}$

- $\arg(z^n) \equiv n\arg(z) \pmod{2\pi}$ for any non-zero complex number z.


Proof

These properties are most easily proven using Euler's form. The proofs are left as exercises.

G GEOMETRY IN THE COORDINATE PLANE

Proposition Addition and Subtraction

Adding or subtracting complex numbers is equivalent to the vector addition or subtraction of their corresponding position vectors, following the **parallelogram law**.

- **Addition $(z_1 + z_2)$**: The vector for the sum is the main diagonal of the parallelogram formed by the vectors for z_1 and z_2.
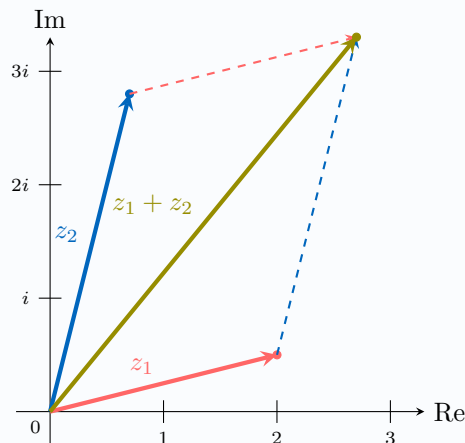
- **Subtraction $(z_1 - z_2)$**: The vector for the difference is the vector from the point representing z_2 to the point representing z_1.
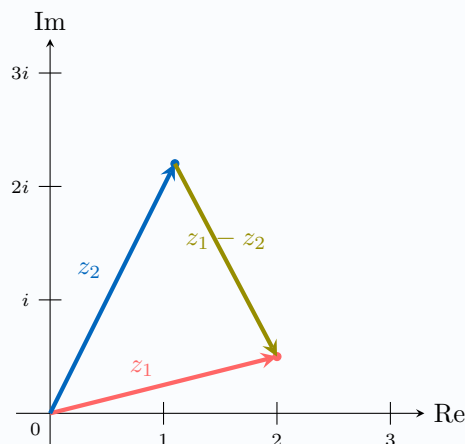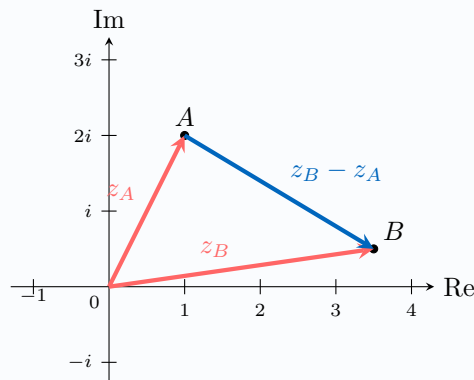
Proposition **Affix of a Vector and Distance between Points**

Let A and B be two points in the complex plane with respective affixes z_A and z_B.

- The vector $\overrightarrow{AB}$ has the affix $z_{\overrightarrow{AB}} = z_B - z_A$.

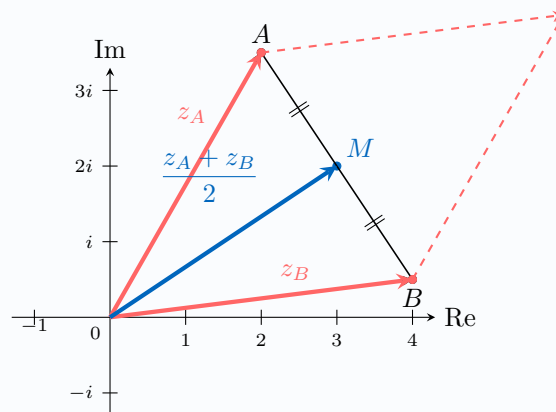- The distance between points A and B is the modulus of this affix:

$$AB = |z_B - z_A|.$$

Proposition **Affix of a Midpoint**

Let A and B be two points in the complex plane with affixes z_A and z_B.
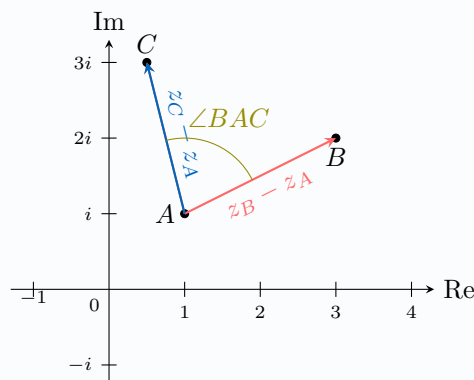The midpoint M of the segment $[AB]$ has the affix

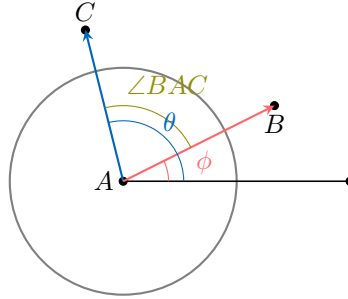$$z_M = \frac{z_A + z_B}{2}.$$

Proposition **Angle**

Let A, B, and C be three distinct points in the complex plane with respective affixes z_A, z_B, and z_C. The measure of the angle $\angle BAC$ is given by

$$\angle BAC = \arg\left(\frac{z_C - z_A}{z_B - z_A}\right).$$

Proof

The measure of the angle $\angle BAC$ is the difference between the angle of the vector $\overrightarrow{AC}$ and the angle of the vector $\overrightarrow{AB}$, both measured from the positive real axis.

Let θ be the angle of the vector $\overrightarrow{AC}$ and ϕ be the angle of the vector $\overrightarrow{AB}$. From the definition of the argument of the affix of a vector, we have:

- The affix of $\overrightarrow{AC}$ is $z_C - z_A$, so $\theta = \arg(z_C - z_A)$.

- The affix of $\overrightarrow{AB}$ is $z_B - z_A$, so $\phi = \arg(z_B - z_A)$.

The angle of the geometric configuration is the difference between these two arguments:

$$\angle BAC = \theta - \phi$$
$$= \arg(z_C - z_A) - \arg(z_B - z_A).$$

Using the property for the argument of a quotient, $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$, we get:

$$\angle BAC = \arg\left(\frac{z_C - z_A}{z_B - z_A}\right).$$

This holds modulo 2π.

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Proposition **Multiplication and Division**

Multiplying or dividing complex numbers corresponds to scaling their moduli and rotating their arguments.

- **Multiplication** ($z_1 z_2$): Multiply the moduli and add the arguments.

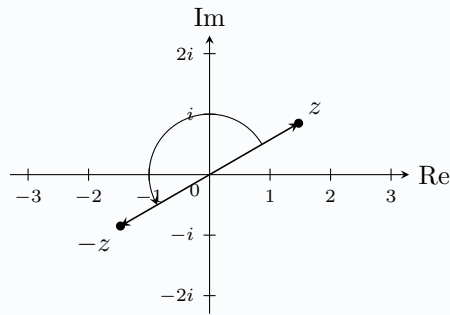- **Division** ($\frac{z_1}{z_2}$): Divide the moduli and subtract the arguments.

More precisely, multiplying a complex number z by $w = re^{i\phi}$ results in scaling the modulus of z by a factor of r and rotating it by an angle of ϕ.
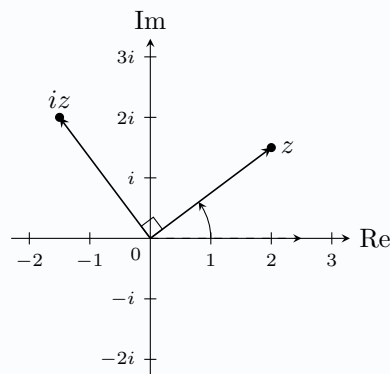

Proposition **Fundamental Geometric Transformations**

- **Conjugate** ($z \to \overline{z}$): a **reflection** across the real axis.
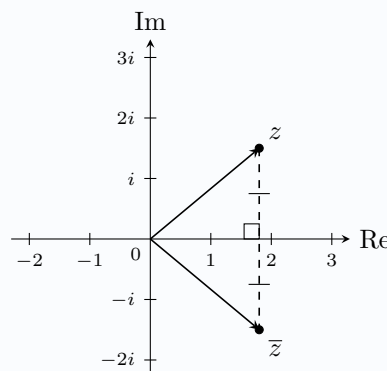
- **Negation** ($z \to -z$): a **rotation** of 180° (or π radians) about the origin.

- **Multiplication by** i $(z \to iz)$: a **rotation** of $90°$ (or $\pi/2$ radians) anti-clockwise about the origin.


H GEOMETRIC LOCI IN THE COMPLEX PLANE

Equations and inequalities involving complex numbers can define geometric shapes or regions, known as loci, in the complex plane. Understanding the geometric meaning of modulus and argument is key to interpreting these loci.

Proposition **Fundamental Geometric Loci**

- **Circles**: The equation $|z - c| = r$ describes a circle with centre c and radius r. This is the set of all points z that are at a fixed distance r from the point c.
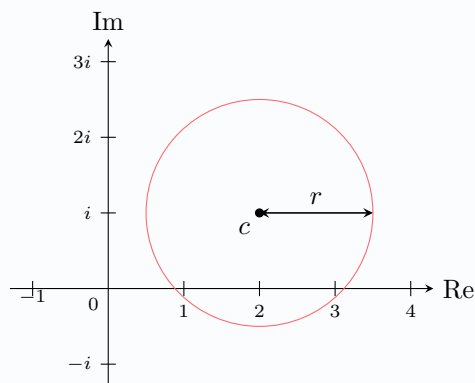
- **Perpendicular Bisectors**: The equation $|z - a| = |z - b|$ describes the perpendicular bisector of the line segment connecting points a and b. This is the set of all points z that are equidistant from a and b.
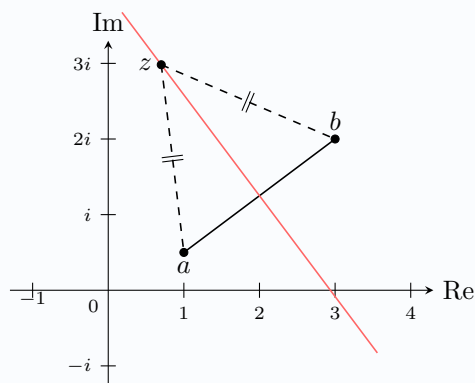
- **Rays**: The equation $\arg(z - c) = \alpha$ describes a ray originating from the point c (but not including c) at a fixed angle α to the positive real direction.
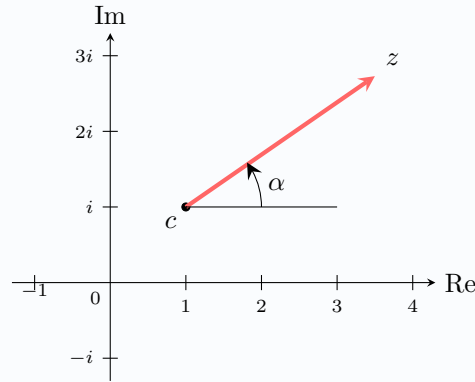
Inequalities describe regions. For example, $|z - c| < r$ represents the interior of a circle (an open disk), and $|z - c| \geq r$ represents the circle together with its exterior.

I ROOTS OF COMPLEX NUMBERS

Just as De Moivre's theorem provides a powerful method for raising complex numbers to a power, it also gives us a systematic way to find the n-th roots of any complex number. We begin with the special case of finding the roots of 1.


Definition Roots of Unity

For any integer $n \geq 1$, an **n-th root of unity** is any complex number z such that $z^n = 1$. The set of all n-th roots of unity is denoted by $\mathbb{U}_n$.

Proposition Structure of the Set of Roots of Unity

For any integer $n \geq 1$, the set of the n-th roots of unity is

$$\mathbb{U}_n = \left\{ e^{i\frac{2k\pi}{n}} \mid k = 0, 1, 2, \ldots, n-1 \right\}.$$

Geometrically, the points representing the n-th roots of unity form the vertices of a regular n-sided polygon inscribed in the unit circle, with one vertex at the point 1.

Equilateral Triangle ($\mathbb{U}_3$)   Square ($\mathbb{U}_4$)   Regular Pentagon ($\mathbb{U}_5$)
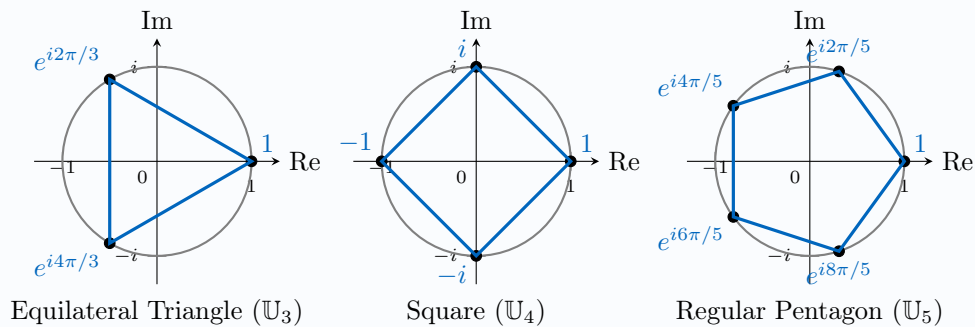
Proposition Finding n-th Roots

A non-zero complex number $c = \rho e^{i\alpha}$ has exactly n distinct n-th roots. The roots are the solutions to the equation $z^n = c$, given by the formula

$$z_k = \rho^{\frac{1}{n}} e^{i\left(\frac{\alpha + 2k\pi}{n}\right)} \quad \text{for } k = 0, 1, 2, \ldots, n-1.$$

Geometrically, the roots form the vertices of a regular n-sided polygon inscribed in a circle of radius $\rho^{1/n}$.

Proof

We want to solve $z^n = c$ for z. Let $z = re^{i\theta}$ and $c = \rho e^{i\alpha}$:

$$(re^{i\theta})^n = \rho e^{i\alpha}$$
$$r^n e^{in\theta} = \rho e^{i\alpha}.$$

For two complex numbers to be equal, their moduli must be equal and their arguments must be coterminal (that is,

differ by a multiple of 2π).

- **Equating moduli:** $r^n = \rho \implies r = \rho^{1/n}$ (since r is a positive real).

- **Equating arguments:** $n\theta = \alpha + 2k\pi \implies \theta = \dfrac{\alpha + 2k\pi}{n}$ for $k \in \mathbb{Z}$.

Substituting these back gives the general solution. We obtain n distinct roots for $k = 0, 1, \ldots, n-1$.

Ex: Find the three cube roots of $c = 8i$.

Answer:

- **Step 1: Convert to Euler form.**
  $\rho = |8i| = 8$. $\alpha = \arg(8i) = \frac{\pi}{2}$. So, $c = 8e^{i\frac{\pi}{2}}$.

- **Step 2: Apply the root formula.**
  The modulus of the roots is $r = 8^{1/3} = 2$.
  The arguments are $\phi_k = \frac{\pi/2 + 2k\pi}{3}$ for $k = 0, 1, 2$.

- **Step 3: Calculate each root.**
  $k = 0 : z_0 = 2e^{i\frac{\pi}{6}} = \sqrt{3} + i$.
  $k = 1 : z_1 = 2e^{i\frac{5\pi}{6}} = -\sqrt{3} + i$.
  $k = 2 : z_2 = 2e^{i\frac{3\pi}{2}} = -2i$.

The roots form an equilateral triangle on a circle of radius 2.