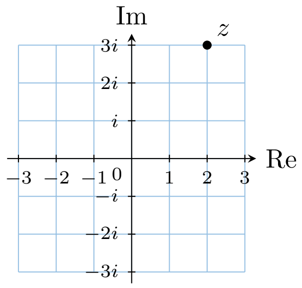
COMPLEX NUMBERS: GEOMETRICAL APPROACH

A COMPLEX PLANE

A.1 READING THE AFFIX OF A POINT

Ex 1:

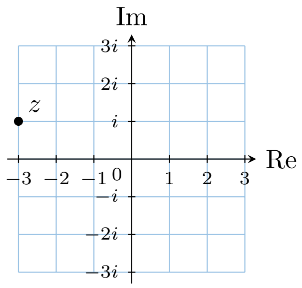


Find the components of z:

$$z = \boxed{2+3i}$$

Answer: The point corresponding to the complex number z has coordinates $(2,3)$ on the complex plane.
The real part of z is the coordinate on the real (horizontal) axis, and the imaginary part is the coordinate on the imaginary (vertical) axis.
Therefore, $\text{Re}(z) = 2$ and $\text{Im}(z) = 3$.
In standard form, this gives $z = 2 + 3i$.

Ex 2:

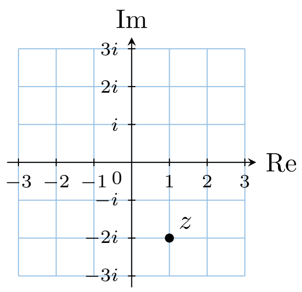


Find the components of z:

$$z = \boxed{-3+i}$$

Answer: The point corresponding to the complex number z has coordinates $(-3,1)$ on the complex plane.
The real part of z is the coordinate on the real (horizontal) axis, and the imaginary part is the coordinate on the imaginary (vertical) axis.
Therefore, $\text{Re}(z) = -3$ and $\text{Im}(z) = 1$.
In standard form, this gives $z = -3 + i$.

Ex 3:

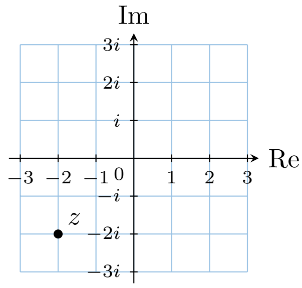


Find the components of z:

$$z = \boxed{1-2i}$$

Answer: The point corresponding to the complex number z has coordinates $(1,-2)$ on the complex plane.
The real part of z is the coordinate on the real (horizontal) axis, and the imaginary part is the coordinate on the imaginary (vertical) axis.
Therefore, $\text{Re}(z) = 1$ and $\text{Im}(z) = -2$.
In standard form, this gives $z = 1 - 2i$.

Ex 4:



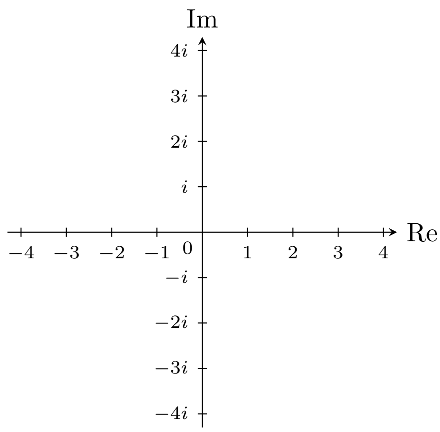
Find the components of z:

$$z = \boxed{-2-2i}$$

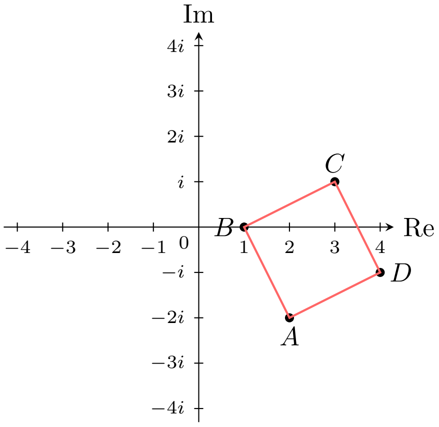
Answer: The point corresponding to the complex number z has coordinates $(-2,-2)$ on the complex plane.
The real part of z is the coordinate on the real (horizontal) axis, and the imaginary part is the coordinate on the imaginary (vertical) axis.
Therefore, $\text{Re}(z) = -2$ and $\text{Im}(z) = -2$.
In standard form, this gives $z = -2 - 2i$.

A.2 CONJECTURING THE NATURE OF A FIGURE

Ex 5: Plot the points A, B, C, and D with respective affixes $z_A = 2-2i$, $z_B = 1$, $z_C = 3+i$, and $z_D = 4-i$.
Conjecture the nature of the quadrilateral $ABCD$.

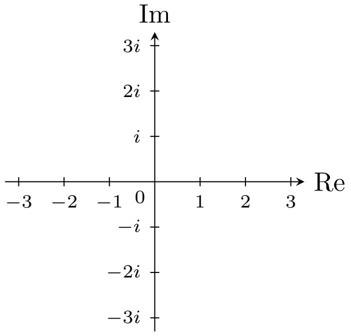


Answer: First, we plot the points corresponding to the affixes: $A(2,-2), B(1,0), C(3,1)$, and $D(4,-1)$.

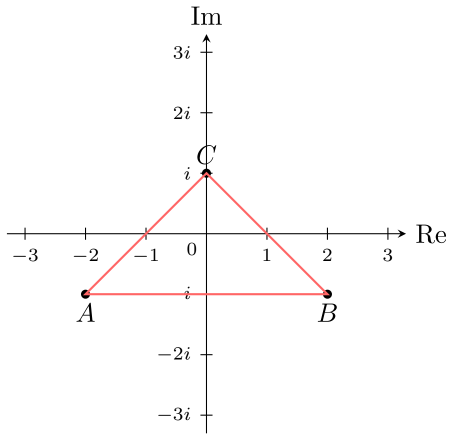


By visual inspection, the quadrilateral $ABCD$ appears to have four equal sides and four right angles. We can therefore **conjecture** that $ABCD$ is a **square**.

Ex 6: Plot the points A, B, and C with respective affixes $z_A = -2 - i$, $z_B = 2 - i$, and $z_C = i$.
Conjecture the nature of the triangle ABC.

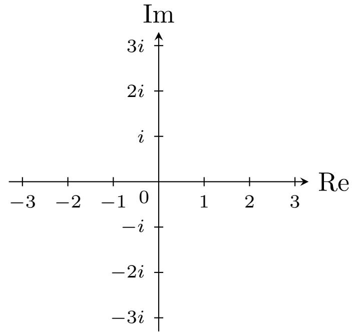


Answer: First, we plot the points corresponding to the affixes: $A(-2, -1)$, $B(2, -1)$, and $C(0, 1)$.

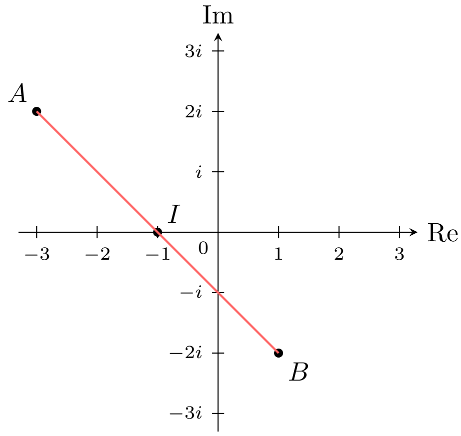


By visual inspection, the triangle ABC appears to have two sides of equal length (AC and BC). We can therefore **conjecture** that ABC is an **isosceles triangle** with the principal vertex C.

Ex 7: Plot the points A, B, and I with respective affixes $z_A = -3 + 2i$, $z_B = 1 - 2i$, and $z_I = -1$.
Conjecture the geometric relationship between the points A, B, and I.



Answer: First, we plot the points corresponding to the affixes: $A(-3, 2)$, $B(1, -2)$, and $I(-1, 0)$.



By visual inspection, the point I appears to lie exactly in the middle of the line segment $\overline{AB}$. We can therefore **conjecture** that I is the **midpoint** of the segment $\overline{AB}$.

B MODULUS AND ARGUMENT

B.1 CALCULATING THE MODULUS OF A COMPLEX NUMBER

Ex 8: Calculate the modulus of the complex number $z = 3 + 4i$.

$$|z| = \boxed{5}$$

Answer: Let $z = x + yi$. The modulus is given by the formula $|z| = \sqrt{x^2 + y^2}$.
For the complex number $z = 3 + 4i$, we have $x = 3$ and $y = 4$.
Substituting these values into the formula:

$$\begin{aligned} |z| &= \sqrt{3^2 + 4^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

The modulus of $z = 3 + 4i$ is 5.

Ex 9: Calculate the modulus of the complex number $z = -1 + 2i$.

$$|z| = \boxed{\sqrt{5}}$$

Answer: Let $z = x + yi$. The modulus is given by the formula $|z| = \sqrt{x^2 + y^2}$.
For the complex number $z = -1 + 2i$, we have $x = -1$ and $y = 2$.
Substituting these values into the formula:

$$\begin{aligned} |z| &= \sqrt{(-1)^2 + 2^2} \\ &= \sqrt{1 + 4} \\ &= \sqrt{5} \end{aligned}$$

The modulus of $z = -1 + 2i$ is $\sqrt{5}$.

Ex 10: Calculate the modulus of the complex number $z = -5i$.

$$|z| = \boxed{5}$$

Answer: Let $z = x + yi$.
The modulus is given by the formula $|z| = \sqrt{x^2 + y^2}$. The complex number $z = -5i$ can be written as $z = 0 - 5i$. We have $x = 0$ and $y = -5$. Substituting these values into the formula:

$$\begin{aligned} |z| &= \sqrt{0^2 + (-5)^2} \\ &= \sqrt{0 + 25} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

The modulus of $z = -5i$ is 5.

Ex 11: Calculate the modulus of the complex number $z = 1 - i$.

$$|z| = \boxed{\sqrt{2}}$$

Answer: Let $z = x + yi$. The modulus is given by the formula $|z| = \sqrt{x^2 + y^2}$.
For the complex number $z = 1 - i$, we have $x = 1$ and $y = -1$.
Substituting these values into the formula:

$$\begin{aligned} |z| &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{1 + 1} \\ &= \sqrt{2} \end{aligned}$$

The modulus of $z = 1 - i$ is $\sqrt{2}$.

B.2 CALCULATING THE ARGUMENT OF A COMPLEX NUMBER

Ex 12: Find the principal argument of the complex number $z = -1 + i\sqrt{3}$. (i.e., the argument in the interval $(-\pi, \pi]$).

$$\arg(z) = \boxed{\frac{2\pi}{3}}$$

Answer:

1. We first calculate the modulus of z.

$$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = 2.$$

We can then factor out the modulus:

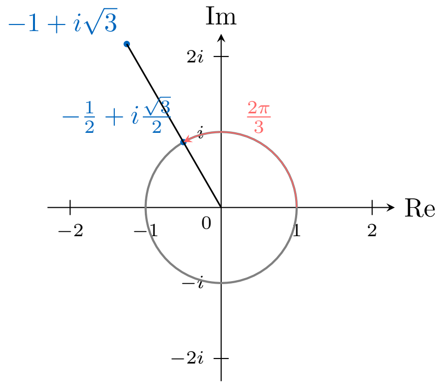
$$z = 2 \times \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

2. We are looking for a real number θ such that:

$$\begin{cases} \cos(\theta) = -\frac{1}{2} \\ \sin(\theta) = \frac{\sqrt{3}}{2} \end{cases}$$

The angle $\theta = \frac{2\pi}{3}$ satisfies both equations. Therefore, an argument of z is $\arg(z) = \frac{2\pi}{3} \pmod{2\pi}$.

3.



Ex 13: Find the principal argument of the complex number $z = 1 - i$. (i.e., the argument in the interval $(-\pi, \pi]$).

$$\arg(z) = \boxed{-\frac{\pi}{4}}$$

Answer:

1. We first calculate the modulus of z.

$$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{1 + 1} = \sqrt{2}.$$

We can then factor out the modulus:

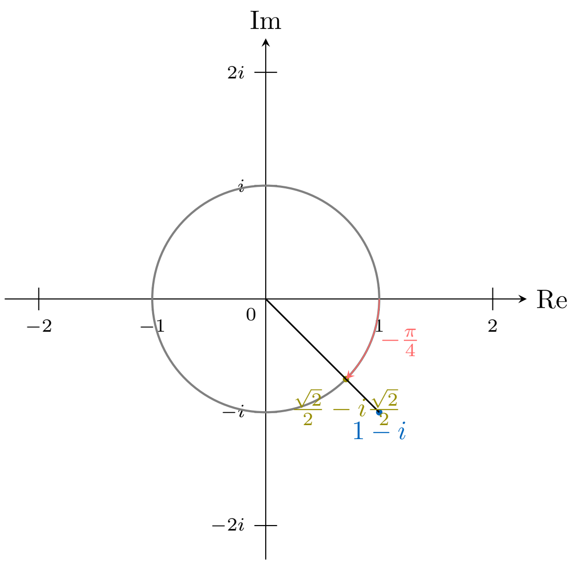
$$z = \sqrt{2} \times \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2} \times \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

2. We are looking for a real number θ such that:

$$\begin{cases} \cos(\theta) = \frac{\sqrt{2}}{2} \\ \sin(\theta) = -\frac{\sqrt{2}}{2} \end{cases}$$

The angle $\theta = -\frac{\pi}{4}$ satisfies both equations. Therefore, an argument of z is $\arg(z) = -\frac{\pi}{4} \pmod{2\pi}$.

3.



Ex 14: Find the principal argument of the complex number $z = \frac{3\sqrt{3}}{2} + i\frac{3}{2}$. (i.e., the argument in the interval $(-\pi, \pi]$).

$$\arg(z) = \boxed{\frac{\pi}{6}}$$

Answer:

1. We first calculate the modulus of z.

$$|z| = \sqrt{\left(\frac{3\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{27}{4} + \frac{9}{4}} = \sqrt{\frac{36}{4}} = \sqrt{9} = 3.$$

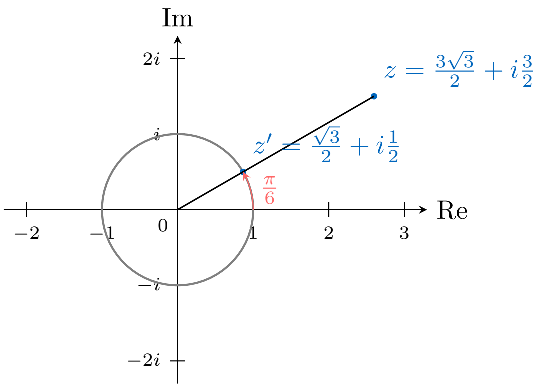
   We can then factor out the modulus:

$$z = 3 \times \left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

2. We are looking for a real number θ such that:

$$\begin{cases} \cos(\theta) = \frac{\sqrt{3}}{2} \\ \sin(\theta) = \frac{1}{2} \end{cases}$$

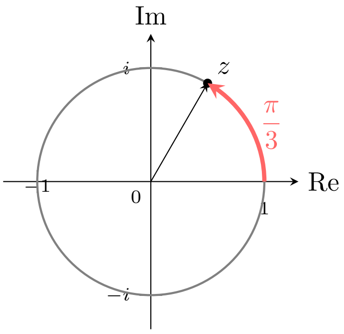
   The angle $\theta = \frac{\pi}{6}$ satisfies both equations. Therefore, an argument of z is $\arg(z) = \frac{\pi}{6} \pmod{2\pi}$.

3. 

C UNIT MODULUS COMPLEX NUMBERS AND THE IMAGINARY EXPONENTIAL

C.1 FINDING THE AFFIX OF A POINT ON THE UNIT CIRCLE

Ex 15:



Find the standard form of the affix z shown in the diagram.

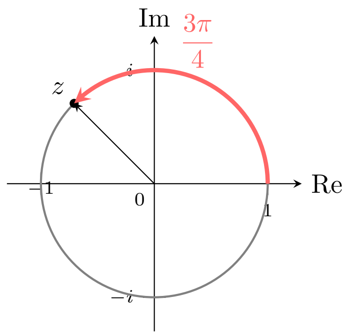
$$z = \boxed{\frac{1}{2} + i\frac{\sqrt{3}}{2}}$$

Answer: The complex number $z = x + yi$ is represented by a point on the unit circle.
By definition of trigonometry on the unit circle, the coordinates of a point at an angle θ from the positive x-axis are given by $(x, y) = (\cos\theta, \sin\theta)$.
In this case, the angle is given as $\theta = \frac{\pi}{3}$.

- The real part is $x = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.
- The imaginary part is $y = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

Therefore, the affix of the point is $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Ex 16:



Find the standard form of the affix z shown in the diagram.

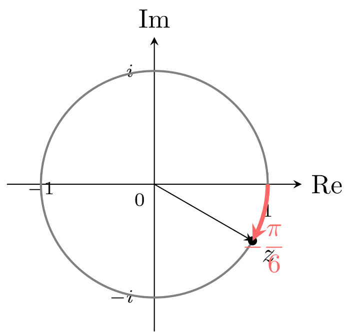
$$z = \boxed{-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}$$

Answer: The complex number $z = x + yi$ is represented by a point on the unit circle.
By definition of trigonometry on the unit circle, the coordinates of a point at an angle θ from the positive x-axis are given by $(x, y) = (\cos\theta, \sin\theta)$.
In this case, the angle is given as $\theta = \frac{3\pi}{4}$.

- The real part is $x = \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.
- The imaginary part is $y = \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

Therefore, the affix of the point is $z = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$.

Ex 17:



Find the standard form of the affix z shown in the diagram.

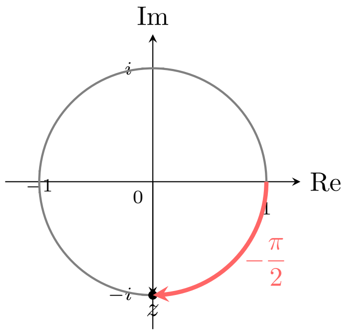
$$z = \boxed{\frac{\sqrt{3}}{2} - i\frac{1}{2}}$$

Answer: The complex number $z = x + yi$ is represented by a point on the unit circle.
By definition of trigonometry on the unit circle, the coordinates of a point at an angle θ from the positive x-axis are given by $(x, y) = (\cos\theta, \sin\theta)$.
In this case, the angle is given as $\theta = -\frac{\pi}{6}$.

- The real part is $x = \cos\left(-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.
- The imaginary part is $y = \sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$.

Therefore, the affix of the point is $z = \frac{\sqrt{3}}{2} - i\frac{1}{2}$.

Ex 18:



Find the standard form of the affix z shown in the diagram.

$$z = \boxed{-i}$$

Answer: The complex number $z = x + yi$ is represented by a point on the unit circle.
By definition of trigonometry on the unit circle, the coordinates of a point at an angle θ from the positive x-axis are given by $(x, y) = (\cos\theta, \sin\theta)$.
In this case, the angle is given as $\theta = -\frac{\pi}{2}$.

- The real part is $x = \cos\left(-\frac{\pi}{2}\right) = 0$.
- The imaginary part is $y = \sin\left(-\frac{\pi}{2}\right) = -1$.

Therefore, the affix of the point is $z = 0 - 1i = -i$.

C.2 EVALUATING COMPLEX EXPONENTIALS

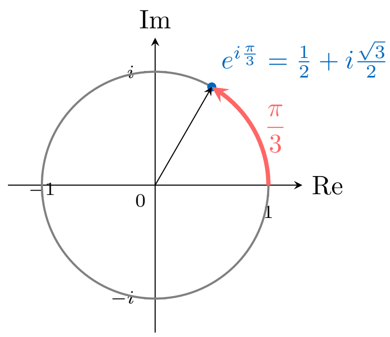
Ex 19: Convert the following complex number to standard form:

$$z = e^{i\frac{\pi}{3}}$$

$$z = \boxed{\frac{1}{2} + i\frac{\sqrt{3}}{2}}$$

Answer: We use Euler's identity, $e^{i\theta} = \cos\theta + i\sin\theta$, to convert the number to polar form, then evaluate the trigonometric functions.
For $z = e^{i\frac{\pi}{3}}$, the angle is $\theta = \frac{\pi}{3}$.

$$\begin{aligned} z &= \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) \\ &= \frac{1}{2} + i\frac{\sqrt{3}}{2} \end{aligned}$$



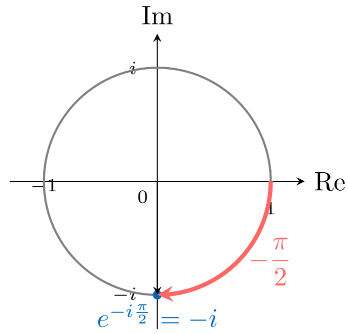
Ex 20: Convert the following complex number to standard form:

$$z = e^{-i\frac{\pi}{2}}$$

$$z = \boxed{-i}$$

Answer: We use Euler's identity, $e^{i\theta} = \cos\theta + i\sin\theta$, to convert the number to polar form, then evaluate the trigonometric functions.
For $z = e^{-i\frac{\pi}{2}}$, the angle is $\theta = -\frac{\pi}{2}$.

$$\begin{aligned} z &= \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) \\ &= 0 + i(-1) \\ &= -i \end{aligned}$$



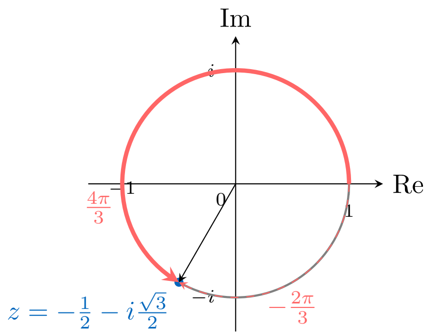
Ex 21: Convert the following complex number to standard form:

$$z = e^{i\frac{4\pi}{3}}$$

$$z = \boxed{-\frac{1}{2} - i\frac{\sqrt{3}}{2}}$$

Answer: We use Euler's identity, $e^{i\theta} = \cos\theta + i\sin\theta$.
The angle is $\theta = \frac{4\pi}{3}$. This angle is in the third quadrant. We can evaluate its cosine and sine directly:

$$\begin{aligned} z &= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) \\ &= -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{aligned}$$



Ex 22: Convert the following complex number to standard form:

$$z = e^{i\frac{13\pi}{6}}$$

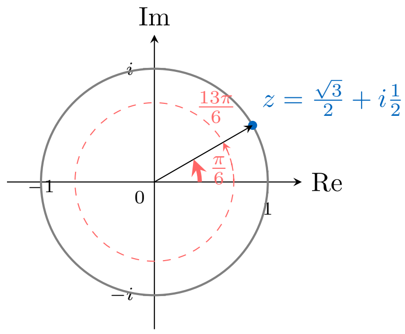
$$z = \boxed{\frac{\sqrt{3}}{2} + i\frac{1}{2}}$$

Answer: We use Euler's identity, $e^{i\theta} = \cos\theta + i\sin\theta$.
The angle is $\theta = \frac{13\pi}{6}$. Since this angle is outside the principal range of $(-\pi, \pi]$, we first find a coterminal angle by subtracting a multiple of 2π.

$$\frac{13\pi}{6} = \frac{12\pi}{6} + \frac{\pi}{6} = 2\pi + \frac{\pi}{6}$$

The position of the point is the same as for the angle $\frac{\pi}{6}$. Therefore, $e^{i\frac{13\pi}{6}} = e^{i\frac{\pi}{6}}$.

$$\begin{aligned} z &= \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{3}}{2} + i\frac{1}{2} \end{aligned}$$



C.3 APPLYING THE PROPERTIES OF EXPONENTS

Ex 23: Given $z_1 = e^{i\frac{2\pi}{3}}$ and $z_2 = e^{i\frac{\pi}{3}}$, calculate and simplify the product z_1z_2.

$$z_1z_2 = \boxed{e^{i\pi}}$$

Answer: We use the laws of exponents to calculate the product.

$$\begin{aligned} z_1z_2 &= e^{i\frac{2\pi}{3}} \times e^{i\frac{\pi}{3}} \\ &= e^{i\frac{2\pi}{3}+i\frac{\pi}{3}} \quad (\text{since } e^ae^b = e^{a+b}) \\ &= e^{i\left(\frac{2\pi}{3}+\frac{\pi}{3}\right)} \\ &= e^{i\frac{3\pi}{3}} \\ &= e^{i\pi} \end{aligned}$$

The product is $z_1z_2 = e^{i\pi}$.

Ex 24: Given $z_1 = e^{i\frac{\pi}{2}}$ and $z_2 = e^{i\frac{\pi}{6}}$, calculate and simplify the quotient $\frac{z_1}{z_2}$.

$$\frac{z_1}{z_2} = \boxed{e^{i\frac{\pi}{3}}}$$

Answer: We use the laws of exponents to calculate the quotient.

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{e^{i\frac{\pi}{2}}}{e^{i\frac{\pi}{6}}} \\ &= e^{i\frac{\pi}{2}-i\frac{\pi}{6}} \quad \left(\text{since } \frac{e^a}{e^b} = e^{a-b}\right) \\ &= e^{i\left(\frac{3\pi}{6}-\frac{\pi}{6}\right)} \\ &= e^{i\frac{2\pi}{6}} \\ &= e^{i\frac{\pi}{3}} \end{aligned}$$

The quotient is $\frac{z_1}{z_2} = e^{i\frac{\pi}{3}}$.

Ex 25: Given $z_1 = e^{i\frac{3\pi}{4}}$ and $z_2 = e^{-i\frac{\pi}{2}}$, calculate and simplify the product z_1z_2.

$$z_1z_2 = \boxed{e^{i\frac{\pi}{4}}}$$

Answer: We use the laws of exponents to calculate the product.

$$\begin{aligned} z_1z_2 &= e^{i\frac{3\pi}{4}} \times e^{-i\frac{\pi}{2}} \\ &= e^{i\frac{3\pi}{4}-i\frac{\pi}{2}} \quad (\text{since } e^ae^b = e^{a+b}) \\ &= e^{i\left(\frac{3\pi}{4}-\frac{2\pi}{4}\right)} \\ &= e^{i\frac{\pi}{4}} \end{aligned}$$

The product is $z_1z_2 = e^{i\frac{\pi}{4}}$. The argument $\frac{\pi}{4}$ is in the interval $]-\pi,\pi]$.

Ex 26: Given $z = e^{i\frac{2\pi}{3}}$, calculate z^3.

$$z^3 = \boxed{e^{i2\pi}}$$

Answer: We use the laws of exponents to calculate the power.

$$\begin{aligned} z^3 &= \left(e^{i\frac{2\pi}{3}}\right)^3 \\ &= e^{i\left(3\times\frac{2\pi}{3}\right)} \quad (\text{since } (e^a)^b = e^{ab}) \\ &= e^{i2\pi} \end{aligned}$$

The result is $z^3 = e^{i2\pi}$, which is equal to 1.

Ex 27: Given $z = e^{i\frac{\pi}{4}}$, find its conjugate $\overline{z}$. Give your answer in Euler's form with a principal argument.

$$\overline{z} = \boxed{e^{-i\frac{\pi}{4}}}$$

Answer: To find the conjugate of a number in Euler's form, we take the conjugate of the expression.

$$\begin{aligned} \overline{z} &= \overline{e^{i\frac{\pi}{4}}} \\ &= \overline{\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)} \\ &= \cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right) \\ &= \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) \quad (\text{since cos is even, sin is odd}) \\ &= e^{-i\frac{\pi}{4}} \end{aligned}$$

The conjugate is $\overline{z} = e^{-i\frac{\pi}{4}}$. The argument $-\frac{\pi}{4}$ is in the interval $]-\pi,\pi]$.

Ex 28: Given $z = e^{i\frac{2\pi}{3}}$, find its conjugate $\overline{z}$. Give your answer in Euler's form with a principal argument.

$$\overline{z} = \boxed{e^{-i\frac{2\pi}{3}}}$$

Answer: We use the definition of the imaginary exponential and the properties of trigonometric functions.

$$\begin{aligned} \overline{z} &= \overline{e^{i\frac{2\pi}{3}}} \\ &= \overline{\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)} \\ &= \cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right) \\ &= \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) \quad (\text{since cos is even, sin is odd}) \\ &= e^{-i\frac{2\pi}{3}} \end{aligned}$$

The conjugate is $\overline{z} = e^{-i\frac{2\pi}{3}}$. The argument $-\frac{2\pi}{3}$ is in the interval $(-\pi,\pi]$.

D POLAR AND EULER'S FORMS

D.1 CONVERTING FROM POLAR TO STANDARD FORM

Ex 29: Convert the following complex number from polar form to standard form:

$$z_1 = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$$

$$z = \boxed{2i}$$

Answer: To convert from polar to standard form, we evaluate the trigonometric functions and distribute the modulus.

$$\begin{aligned} z &= 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) \\ &= 2(0 + i\cdot 1) \\ &= 2i \end{aligned}$$

Ex 30: Convert the following complex number from polar form to standard form:

$$z = 3\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$

$$z = \boxed{-\frac{3}{2} + i\frac{3\sqrt{3}}{2}}$$

Answer: We evaluate the trigonometric functions and distribute the modulus.

$$\begin{aligned} z &= 3\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) \\ &= 3\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= -\frac{3}{2} + i\frac{3\sqrt{3}}{2} \end{aligned}$$

Ex 31: Convert the following complex number from polar form to standard form:

$$z = 4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

$$z = \boxed{2 - 2i\sqrt{3}}$$

Answer: We evaluate the trigonometric functions, remembering that cosine is an even function ($\cos(-x) = \cos(x)$) and sine is an odd function ($\sin(-x) = -\sin(x)$).

$$\begin{aligned} z &= 4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) \\ &= 4\left(\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\right) \\ &= 4\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \\ &= 2 - 2i\sqrt{3} \end{aligned}$$

Ex 32: Convert the following complex number from polar form to standard form:

$$z = \frac{1}{2}\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right)$$

$$z = \boxed{-\frac{\sqrt{3}}{4} - \frac{1}{4}i}$$

Answer: We evaluate the trigonometric functions and distribute the modulus.

$$\begin{aligned} z &= \frac{1}{2}\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right) \\ &= \frac{1}{2}\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) \\ &= -\frac{\sqrt{3}}{4} - \frac{1}{4}i \end{aligned}$$

D.2 CONVERTING FROM STANDARD TO POLAR FORM

Ex 33: Convert the complex number $z = -1 + i\sqrt{3}$ to polar form.

$$z = \boxed{2(\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}))}$$

Answer: To convert to polar form $z = r(\cos\theta + i\sin\theta)$, we must find the modulus r and the argument θ.

1. **Calculate the modulus (r):**
$$r = |z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2.$$
2. **Find the argument (θ):** We factor out the modulus: $z = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$. We look for an angle θ such that $\cos(\theta) = -1/2$ and $\sin(\theta) = \sqrt{3}/2$. This corresponds to $\theta = \frac{2\pi}{3}$.
3. **Write in polar form:**
$$z = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$

Ex 34: Convert the complex number $z = 1 - i$ to polar form.

$$z = \boxed{\sqrt{2}(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}))}$$

Answer: To convert to polar form $z = r(\cos\theta + i\sin\theta)$, we must find the modulus r and the argument θ.

1. **Calculate the modulus (r):**
$$r = |z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$$
2. **Find the argument (θ):** We factor out the modulus: $z = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$. We look for an angle θ such that $\cos(\theta) = \sqrt{2}/2$ and $\sin(\theta) = -\sqrt{2}/2$. This corresponds to the principal argument $\theta = -\frac{\pi}{4}$.
3. **Write in polar form:**
$$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

Ex 35: Convert the complex number $z = \frac{3\sqrt{3}}{2} + i\frac{3}{2}$ to polar form.

$$z = \boxed{3(\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6}))}$$

Answer: To convert to polar form $z = r(\cos\theta + i\sin\theta)$, we must find the modulus r and the argument θ.

1. **Calculate the modulus (r):**
$$r = |z| = \sqrt{\left(\frac{3\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{27}{4} + \frac{9}{4}} = \sqrt{9} = 3.$$
2. **Find the argument (θ):** We factor out the modulus: $z = 3\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$. We look for an angle θ such that $\cos(\theta) = \sqrt{3}/2$ and $\sin(\theta) = 1/2$. This corresponds to $\theta = \frac{\pi}{6}$.
3. **Write in polar form:**
$$z = 3\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$

D.3 CONVERTING FROM POLAR TO EULER'S FORM

Ex 36: Convert the following complex number from polar form to Euler's form:

$$z = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$

$$z = \boxed{2e^{i\frac{2\pi}{3}}}$$

Answer: To convert from polar form $z = r(\cos\theta + i\sin\theta)$ to Euler's form $z = re^{i\theta}$, we simply identify the modulus r and the argument θ.

- By inspection, the modulus is $r = 2$.
- The argument is $\theta = \frac{2\pi}{3}$.

Substituting these into Euler's form gives:

$$z = 2e^{i\frac{2\pi}{3}}$$

Ex 37: Convert the following complex number from polar form to Euler's form:

$$z = 5\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

$$z = \boxed{5e^{-i\frac{\pi}{4}}}$$

Answer: To convert from polar form $z = r(\cos\theta + i\sin\theta)$ to Euler's form $z = re^{i\theta}$, we simply identify the modulus r and the argument θ.

- By inspection, the modulus is $r = 5$.
- The argument is $\theta = -\frac{\pi}{4}$.

Substituting these into Euler's form gives:

$$z = 5e^{-i\frac{\pi}{4}}$$

Ex 38: Convert the following complex number from polar form to Euler's form:

$$z = \sqrt{3}\,(\cos(\pi) + i\sin(\pi))$$

$$z = \boxed{\sqrt{3}e^{i\pi}}$$

Answer: To convert from polar form $z = r(\cos\theta + i\sin\theta)$ to Euler's form $z = re^{i\theta}$, we simply identify the modulus r and the argument θ.

- By inspection, the modulus is $r = \sqrt{3}$.
- The argument is $\theta = \pi$.

Substituting these into Euler's form gives:

$$z = \sqrt{3}e^{i\pi}$$

E DE MOIVRE'S THEOREM

E.1 APPLYING DE MOIVRE'S THEOREM

Ex 39: Write $(1+i)^8$ in standard form.

$$(1+i)^8 = \boxed{16}$$

Answer: To apply De Moivre's theorem, we first convert $z = 1+i$ to polar or Euler's form.

- **Modulus:** $r = |1+i| = \sqrt{1^2+1^2} = \sqrt{2}$.
- **Argument:** $z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$. We identify $\cos\theta = \frac{\sqrt{2}}{2}$ and $\sin\theta = \frac{\sqrt{2}}{2}$, which gives $\theta = \frac{\pi}{4}$.
- **Euler's Form:** $1+i = \sqrt{2}e^{i\frac{\pi}{4}}$.

Now we apply De Moivre's Theorem:

$$\begin{aligned}(1+i)^8 &= \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^8\\ &= (\sqrt{2})^8\left(e^{i\frac{\pi}{4}}\right)^8\\ &= 16\cdot e^{i\left(8\cdot\frac{\pi}{4}\right)}\\ &= 16e^{i2\pi}\\ &= 16(\cos(2\pi) + i\sin(2\pi))\\ &= 16(1+0i)\\ &= 16.\end{aligned}$$

Ex 40: Write $(\sqrt{3}-i)^6$ in standard form.

$$(\sqrt{3}-i)^6 = \boxed{-64}$$

Answer: First, we convert $z = \sqrt{3}-i$ to Euler's form.

- **Modulus:** $r = |\sqrt{3}-i| = \sqrt{(\sqrt{3})^2+(-1)^2} = \sqrt{3+1} = 2$.
- **Argument:** $z = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$. We identify $\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = -\frac{1}{2}$, which gives the principal argument $\theta = -\frac{\pi}{6}$.
- **Euler's Form:** $z = 2e^{-i\frac{\pi}{6}}$.

Now, we apply De Moivre's Theorem:

$$\begin{aligned}(\sqrt{3}-i)^6 &= \left(2e^{-i\frac{\pi}{6}}\right)^6\\ &= 2^6e^{i\left(6\cdot-\frac{\pi}{6}\right)}\\ &= 64e^{-i\pi}\\ &= 64(\cos(-\pi) + i\sin(-\pi))\\ &= 64(-1+0i)\\ &= -64.\end{aligned}$$

Ex 41: Write $(-2-2i)^3$ in standard form.

$$(-2-2i)^3 = \boxed{16-16i}$$

Answer: We use De Moivre's theorem. First, we convert the base $z = -2-2i$ to Euler's form.

- **Modulus:** $r = |-2-2i| = \sqrt{(-2)^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$.

- **Argument:** Factoring out the modulus gives $z = 2\sqrt{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$. We identify $\cos\theta = -\frac{\sqrt{2}}{2}$ and $\sin\theta = -\frac{\sqrt{2}}{2}$, which corresponds to the principal argument $\theta = -\frac{3\pi}{4}$.
- **Euler's Form:** $z = 2\sqrt{2}e^{-i\frac{3\pi}{4}}$.

Now, we apply De Moivre's Theorem:

$$\begin{aligned}(-2-2i)^3 &= \left(2\sqrt{2}e^{-i\frac{3\pi}{4}}\right)^3 = (2\sqrt{2})^3 e^{i\left(3\cdot -\frac{3\pi}{4}\right)}\\ &= 16\sqrt{2}e^{-i\frac{9\pi}{4}}\end{aligned}$$

To convert this to standard form, we analyze the argument. We can decompose it to show the full rotations:

$$-\frac{9\pi}{4} = -\frac{8\pi}{4} - \frac{\pi}{4} = -2\pi - \frac{\pi}{4}$$

Due to the 2π-periodicity of cosine and sine, we have $\cos(-\frac{9\pi}{4}) = \cos(-\frac{\pi}{4})$ and $\sin(-\frac{9\pi}{4}) = \sin(-\frac{\pi}{4})$.

$$\begin{aligned}16\sqrt{2}e^{-i\frac{9\pi}{4}} &= 16\sqrt{2}\left(\cos\left(-\frac{9\pi}{4}\right) + i\sin\left(-\frac{9\pi}{4}\right)\right)\\ &= 16\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)\\ &= 16\sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\\ &= 16(1-i) = 16 - 16i.\end{aligned}$$

F PROPERTIES OF MODULUS AND ARGUMENT

F.1 PROVING THE PROPERTIES OF THE MODULUS

Ex 42: Prove that $|\overline{z}| = |z|$ for any complex number z.

Answer:

- **Proof using Standard Form**
  Let $z = x + yi$.
  $$\begin{aligned}|\overline{z}| &= |x - yi|\\ &= \sqrt{x^2 + (-y)^2}\\ &= \sqrt{x^2+y^2}\\ &= |z|\end{aligned}$$
- **Proof using Euler's Form**
  Let $z = re^{i\theta}$.
  $$\begin{aligned}|\overline{z}| &= |\overline{re^{i\theta}}|\\ &= |\overline{r}\cdot\overline{e^{i\theta}}|\\ &= |re^{-i\theta}|\\ &= |r|\cdot|e^{-i\theta}| \quad (\text{since } |z_1z_2| = |z_1||z_2|)\\ &= r\cdot 1 \quad (\text{since } r \geq 0 \text{ and } |e^{i\phi}| = 1)\\ &= |z|\end{aligned}$$

Ex 43: Prove that $|z|^2 = z\overline{z}$ for any complex number z.

Answer:

- **Proof using Standard Form**
  Let $z = x + yi$.
  $$\begin{aligned}z\overline{z} &= (x+yi)(x-yi)\\ &= x^2 - (iy)^2\\ &= x^2 - i^2y^2\\ &= x^2 + y^2\\ &= \left(\sqrt{x^2+y^2}\right)^2\\ &= |z|^2\end{aligned}$$
- **Proof using Euler's Form**
  Let $z = re^{i\theta}$.
  $$\begin{aligned}z\overline{z} &= (re^{i\theta})(re^{-i\theta})\\ &= r^2e^{i\theta - i\theta}\\ &= r^2e^0\\ &= r^2\\ &= |z|^2\end{aligned}$$

Ex 44: Prove that $|z_1z_2| = |z_1||z_2|$ for any complex numbers z_1 and z_2.

Answer:

- **Proof using the property $|z|^2 = z\overline{z}$**
  We start by squaring the left-hand side:
  $$\begin{aligned}|z_1z_2|^2 &= (z_1z_2)(\overline{z_1z_2}) \quad (\text{using } |w|^2 = w\overline{w})\\ &= z_1z_2\overline{z_1}\,\overline{z_2} \quad (\text{since } \overline{ab} = \overline{a}\overline{b})\\ &= (z_1\overline{z_1})(z_2\overline{z_2})\\ &= |z_1|^2|z_2|^2\\ &= (|z_1||z_2|)^2\end{aligned}$$
  Since $|z_1z_2|$ and $|z_1||z_2|$ are non-negative, taking the square root gives $|z_1z_2| = |z_1||z_2|$.
- **Proof using Euler's Form**
  Let $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$.
  $$\begin{aligned}|z_1z_2| &= |(r_1e^{i\theta_1})(r_2e^{i\theta_2})|\\ &= |(r_1r_2)e^{i(\theta_1+\theta_2)}|\\ &= r_1r_2 \quad (\text{by definition of modulus in Euler's form})\\ &= |z_1||z_2|\end{aligned}$$

F.2 PROVING THE PROPERTIES OF THE ARGUMENT

Ex 45: Prove that $\arg(\overline{z}) = -\arg(z) \pmod{2\pi}$ for any non-zero complex number z.

Answer: Let $z = re^{i\theta}$.

$$\begin{aligned}\arg(\overline{z}) &= \arg(\overline{re^{i\theta}})\\ &= \arg(\overline{r}\cdot\overline{e^{i\theta}})\\ &= \arg(re^{-i\theta})\\ &= -\theta \pmod{2\pi}\\ &= -\arg(z) \pmod{2\pi}\end{aligned}$$

Ex 46: Prove that $\arg(z_1z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi}$ for any non-zero complex numbers z_1 and z_2.

Answer: Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.

$$\begin{aligned}
\arg(z_1 z_2) &= \arg\left((r_1 e^{i\theta_1})(r_2 e^{i\theta_2})\right) \\
&= \arg\left((r_1 r_2) e^{i(\theta_1+\theta_2)}\right) \\
&= \theta_1 + \theta_2 \pmod{2\pi} \\
&= \arg(z_1) + \arg(z_2) \pmod{2\pi}
\end{aligned}$$

Ex 47: Prove that $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \pmod{2\pi}$ for any non-zero complex numbers z_1 and z_2.

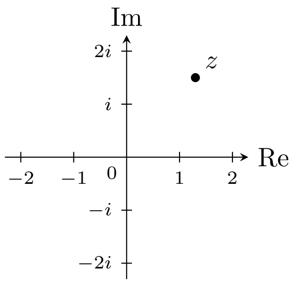
Answer: Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.

$$\begin{aligned}
\arg\left(\frac{z_1}{z_2}\right) &= \arg\left(\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right) \\
&= \arg\left(\left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)}\right) \\
&= \theta_1 - \theta_2 \pmod{2\pi} \\
&= \arg(z_1) - \arg(z_2) \pmod{2\pi}
\end{aligned}$$

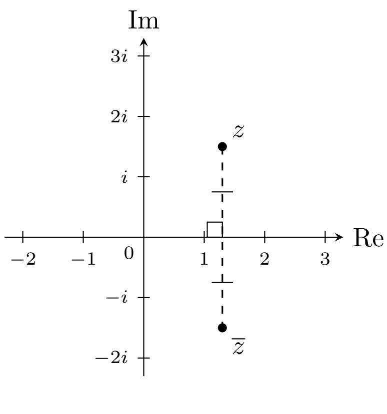
G GEOMETRY IN THE COORDINATE PLANE

G.1 VISUALIZING FUNDAMENTAL TRANSFORMATIONS

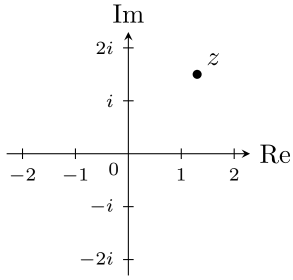
Ex 48: Given the point with affix z below, plot the point with affix $\overline{z}$.



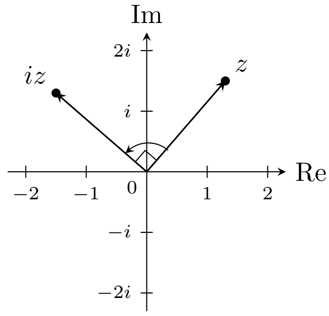
Answer: Let $z = x + yi$. The conjugate is $\overline{z} = x - yi$.
The point for z has coordinates (x, y), while the point for $\overline{z}$ has coordinates $(x, -y)$.
Geometrically, this corresponds to a **reflection** across the real axis (the x-axis).



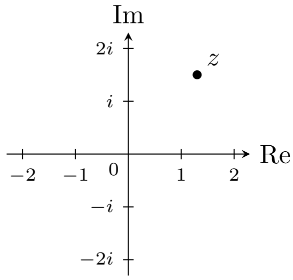
Ex 49: Given the point with affix z below, plot the point with affix iz.



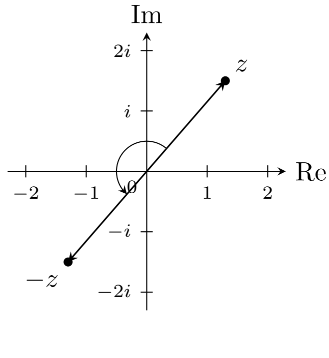
Answer: Let $z = x + yi$. Multiplying by i gives $iz = i(x + yi) = ix + i^2 y = -y + ix$.
The point for z has coordinates (x, y), while the point for iz has coordinates $(-y, x)$.
Geometrically, this corresponds to a **rotation** of $+90°$ (or $+\frac{\pi}{2}$ radians) anti-clockwise about the origin.



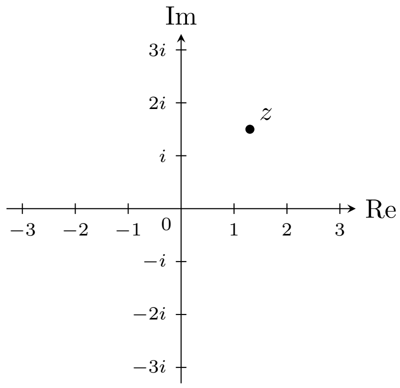
Ex 50: Given the point with affix z below, plot the point with affix $-z$.



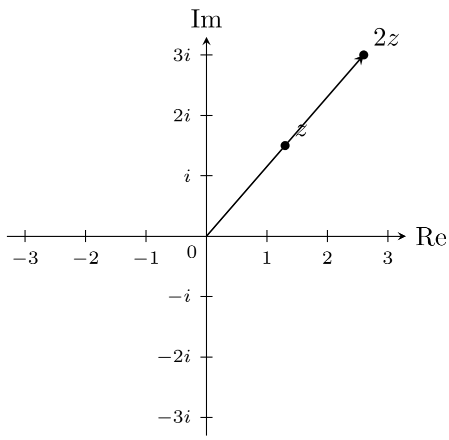
Answer: Let $z = x+yi$. The negative is $-z = -(x+yi) = -x-yi$.
The point for z has coordinates (x, y), while the point for $-z$ has coordinates $(-x, -y)$.
Geometrically, this corresponds to a **rotation** of $180°$ (or π radians) about the origin. This is also equivalent to a point reflection through the origin.



Ex 51: Given the point with affix z below, plot the point with affix $2z$.



Answer: Let $z = x + yi$. Multiplying by 2 gives $2z = 2(x + yi) = 2x + 2yi$.
The point for z has coordinates (x, y), while the point for $2z$ has coordinates $(2x, 2y)$.
Geometrically, this corresponds to a **dilation** (scaling) centered at the origin with a scale factor of 2. The point $2z$ lies on the same ray from the origin as z, but is twice as far away.



G.2 CALCULATING DISTANCES, MIDPOINTS, AND ANGLES

Ex 52: Given the points $A(2, 3)$ and $B(6, 1)$ on the Cartesian plane. Use complex numbers to find:

1. the distance AB
2. the midpoint of the segment $[AB]$.

Answer: We associate the points A and B with their respective affixes, $z_A = 2 + 3i$ and $z_B = 6 + i$.

1. The distance AB is given by $|z_B - z_A|$.

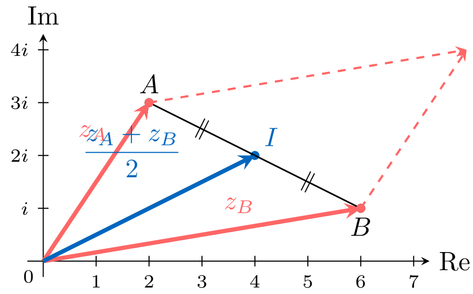
$$\begin{aligned} |z_B - z_A| &= |(6 + i) - (2 + 3i)| \\ &= |4 - 2i| \\ &= \sqrt{4^2 + (-2)^2} \\ &= \sqrt{16 + 4} = \sqrt{20} \end{aligned}$$

   The distance AB is $\sqrt{20}$ units.

2. The affix of the midpoint of the segment $[AB]$ is $\frac{z_A + z_B}{2}$.

$$\begin{aligned} \frac{z_A + z_B}{2} &= \frac{(2 + 3i) + (6 + i)}{2} \\ &= \frac{8 + 4i}{2} \\ &= 4 + 2i \end{aligned}$$

   Therefore, the midpoint of $[AB]$ is the point with coordinates $(4, 2)$.



Ex 53: Given the points $A(-1, 5)$ and $B(3, -1)$ on the Cartesian plane. Use complex numbers to find:

1. the distance AB
2. the midpoint of the segment $[AB]$.

Answer: We associate the points A and B with their respective affixes, $z_A = -1 + 5i$ and $z_B = 3 - i$.

1. The distance AB is given by $|z_B - z_A|$.

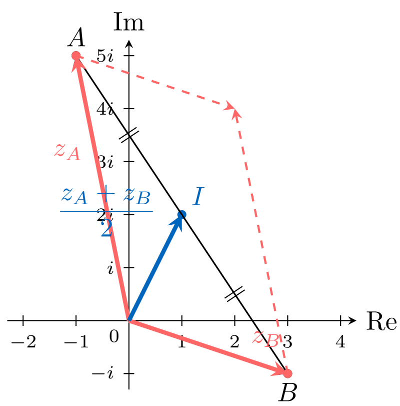
$$\begin{aligned} |z_B - z_A| &= |(3 - i) - (-1 + 5i)| \\ &= |3 - i + 1 - 5i| \\ &= |4 - 6i| \\ &= \sqrt{4^2 + (-6)^2} \\ &= \sqrt{16 + 36} = \sqrt{52} \end{aligned}$$

   The distance AB is $\sqrt{52}$ units.

2. The affix of the midpoint of the segment $[AB]$ is $\frac{z_A + z_B}{2}$.

$$\begin{aligned} \frac{z_A + z_B}{2} &= \frac{(-1 + 5i) + (3 - i)}{2} \\ &= \frac{2 + 4i}{2} \\ &= 1 + 2i \end{aligned}$$

   Therefore, the midpoint of $[AB]$ is the point with coordinates $(1, 2)$.



Ex 54: Let A, B, and C be three points in the complex plane with respective affixes $z_A = 1$, $z_B = 3$, and $z_C = 3 + 2i\sqrt{3}$. Calculate the measure of the angle $\angle BAC$.

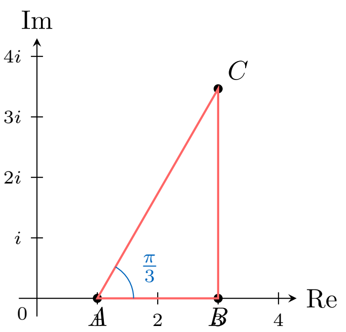
$$\angle BAC = \boxed{\frac{\pi}{3}}$$

Answer: The measure of the angle $\angle BAC$ is given by the formula:

$$\begin{aligned}\angle BAC &= \arg\left(\frac{z_C - z_A}{z_B - z_A}\right)\\ &= \arg\left(\frac{(3+2i\sqrt{3})-1}{3-1}\right)\\ &= \arg\left(\frac{2+2i\sqrt{3}}{2}\right)\\ &= \arg\left(1+i\sqrt{3}\right)\end{aligned}$$

To find the argument of $1+i\sqrt{3}$, we can convert it to polar form.
The modulus is $|1+i\sqrt{3}| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$.
So, $1+i\sqrt{3} = 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$.
The argument is therefore $\frac{\pi}{3}$.

$$\angle BAC = \frac{\pi}{3}$$



Ex 55: Let A, B, and C be three points in the complex plane with respective affixes $z_A = 1+i$, $z_B = -1+3i$, and $z_C = 2+2i$. Calculate the measure of the angle $\angle BAC$.

$$\angle BAC = \boxed{-\frac{\pi}{2}}$$

Answer: The measure of the angle $\angle BAC$ is given by the argument of the quotient $\frac{z_C - z_A}{z_B - z_A}$.

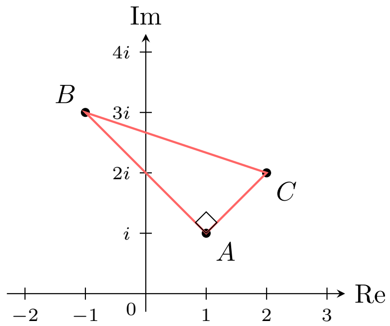
$$\begin{aligned}\angle BAC &= \arg\left(\frac{z_C - z_A}{z_B - z_A}\right)\\ &= \arg\left(\frac{(2+2i)-(1+i)}{(-1+3i)-(1+i)}\right)\\ &= \arg\left(\frac{1+i}{-2+2i}\right)\end{aligned}$$

To find the argument, we convert the quotient to standard form:

$$\begin{aligned}\frac{1+i}{-2+2i} &= \frac{1+i}{2(-1+i)} = \frac{(1+i)(-1-i)}{2(-1+i)(-1-i)}\\ &= \frac{-1-i-i-i^2}{2(1+1)} = \frac{-1-2i+1}{4} = \frac{-2i}{4} = -\frac{1}{2}i\end{aligned}$$

The angle is the argument of this result:

$$\angle BAC = \arg\left(-\frac{1}{2}i\right) = \arg\left(-\frac{1}{2}\left(\cos(-\frac{\pi}{2})+i\sin(-\frac{\pi}{2})\right)\right) = -\frac{\pi}{2}$$



G.3 PROVING THE NATURE OF GEOMETRIC FIGURES

Ex 56: Let A, B, and C be three points in the complex plane with respective affixes $z_A = 1+2i$, $z_B = 3+3i$, and $z_C = 2$. Prove that the triangle ABC is isosceles at A.

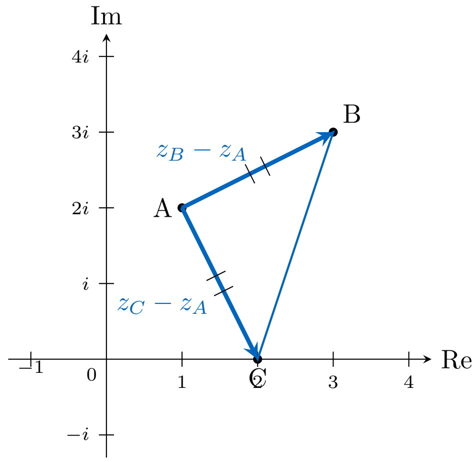
Answer: To prove that triangle ABC is isosceles at A, we must show that the lengths of the sides AB and AC are equal. In the language of complex numbers, this means we must prove that $|z_B - z_A| = |z_C - z_A|$.
First, we calculate the length of the side AB:

$$\begin{aligned}AB &= |z_B - z_A|\\ &= |(3+3i)-(1+2i)|\\ &= |3-1+3i-2i|\\ &= |2+i|\\ &= \sqrt{2^2+1^2}\\ &= \sqrt{4+1} = \sqrt{5}\end{aligned}$$

Next, we calculate the length of the side AC:

$$\begin{aligned}AC &= |z_C - z_A|\\ &= |2-(1+2i)|\\ &= |2-1-2i|\\ &= |1-2i|\\ &= \sqrt{1^2+(-2)^2}\\ &= \sqrt{1+4} = \sqrt{5}\end{aligned}$$

Since $|z_B - z_A| = |z_C - z_A| = \sqrt{5}$, the lengths AB and AC are equal.
Therefore, triangle ABC is isosceles at A.



Ex 57: Let A, B, and C be three points in the complex plane with respective affixes $z_A = 1+i$, $z_B = 3+2i$, and $z_C = 2+4i$.

Prove that the triangle ABC is a right-angled isosceles triangle at B.

Answer: To prove that triangle ABC is a right-angled isosceles triangle at B, we need to show two things:

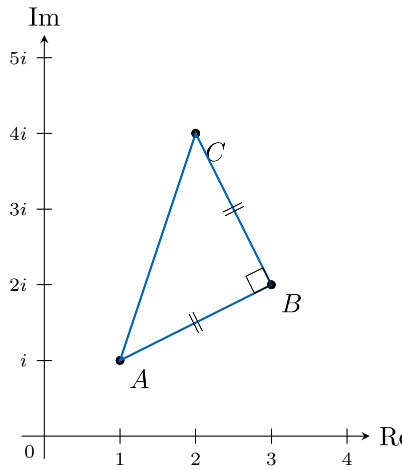
1. The lengths of the sides BA and BC are equal ($|z_A - z_B| = |z_C - z_B|$).
2. The angle at B is a right angle ($\arg\left(\frac{z_A - z_B}{z_C - z_B}\right) = \pm\frac{\pi}{2}$).

First, we calculate the affixes of the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$:

$$z_{\overrightarrow{BA}} = z_A - z_B = (1+i) - (3+2i) = -2 - i$$
$$z_{\overrightarrow{BC}} = z_C - z_B = (2+4i) - (3+2i) = -1 + 2i$$

- **Check lengths (moduli):**
  $$BA = |-2-i| = \sqrt{(-2)^2 + (-1)^2} = \sqrt{4+1} = \sqrt{5}$$
  $$BC = |-1+2i| = \sqrt{(-1)^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$
  Since the lengths are equal, the triangle is isosceles at B.
- **Check angle (argument of ratio):** The angle at B, $\angle CBA$, is given by $\arg\left(\frac{z_A - z_B}{z_C - z_B}\right)$.
  $$\frac{z_A - z_B}{z_C - z_B} = \frac{-2-i}{-1+2i} = \frac{(-2-i)(-1-2i)}{(-1+2i)(-1-2i)}$$
  $$= \frac{2+4i+i+2i^2}{(-1)^2+2^2} = \frac{2+5i-2}{5} = \frac{5i}{5} = i$$
  The argument of this ratio is $\arg(i) = \frac{\pi}{2}$.

Since the triangle is isosceles at B and the angle at B is $\frac{\pi}{2}$ (a right angle), triangle ABC is a right-angled isosceles triangle.

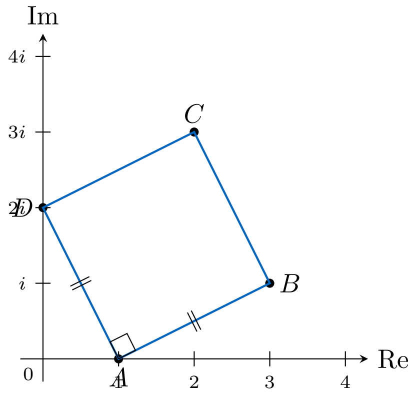


Ex 58: Let A, B, C, and D be four points in the complex plane with respective affixes $z_A = 1$, $z_B = 3+i$, $z_C = 2+3i$, and $z_D = 2i$.
Prove that the quadrilateral ABCD is a square.

Answer: To prove that ABCD is a square, we can prove that it is a parallelogram with two adjacent sides that are equal in length and perpendicular.

1. **Prove ABCD is a parallelogram.**
   We check if the vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are equal by comparing their affixes.
   $$z_{\overrightarrow{AB}} = z_B - z_A = (3+i) - 1 = 2 + i$$
   $$z_{\overrightarrow{DC}} = z_C - z_D = (2+3i) - (2i) = 2 + i$$
   Since $z_{\overrightarrow{AB}} = z_{\overrightarrow{DC}}$, the quadrilateral ABCD is a parallelogram.
2. **Analyze adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{AD}$.**
   We already have $z_{\overrightarrow{AB}} = 2 + i$. Now we find the affix of $\overrightarrow{AD}$:
   $$z_{\overrightarrow{AD}} = z_D - z_A = 2i - 1 = -1 + 2i$$
   - **Check lengths (moduli):**
     $$AB = |z_{\overrightarrow{AB}}| = |2+i| = \sqrt{2^2+1^2} = \sqrt{5}$$
     $$AD = |z_{\overrightarrow{AD}}| = |-1+2i| = \sqrt{(-1)^2+2^2} = \sqrt{5}$$
     Since adjacent sides are equal, the parallelogram is a rhombus.
   - **Check angle (argument of ratio):** The angle at A, $\angle DAB$, is given by $\arg\left(\frac{z_B - z_A}{z_D - z_A}\right)$.
     $$\frac{z_B - z_A}{z_D - z_A} = \frac{2+i}{-1+2i} = \frac{(2+i)(-1-2i)}{(-1+2i)(-1-2i)}$$
     $$= \frac{-2-4i-i-2i^2}{1+4} = \frac{-2-5i+2}{5} = \frac{-5i}{5} = -i$$
     The argument of this ratio is $\arg(-i) = -\frac{\pi}{2}$.

Since ABCD is a rhombus with a right angle at A, it is a square.

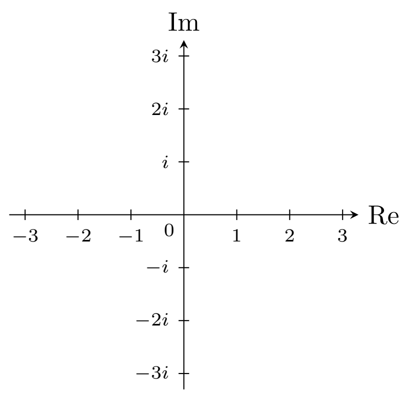


H GEOMETRIC LOCI IN THE COMPLEX PLANE

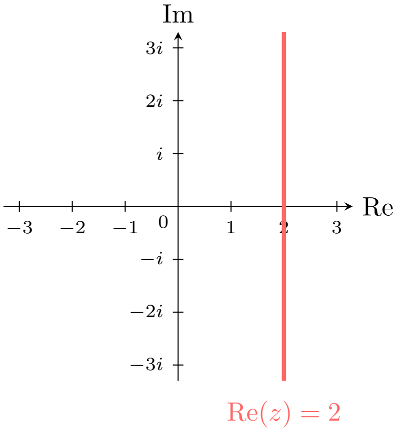
H.1 PLOTTING LINES

Ex 59: Plot the set of points M in the plane whose affix z satisfies:

$$\text{Re}(z) = 2$$

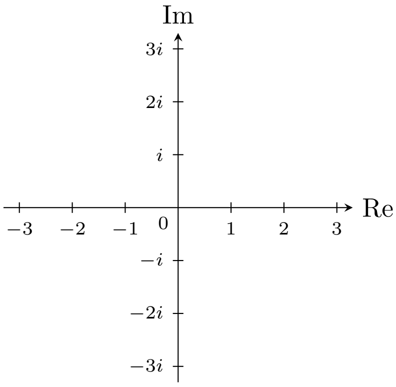


Answer: Let $z = x + yi$. The real part of z is $\text{Re}(z) = x$.
The condition $\text{Re}(z) = 2$ is therefore equivalent to the equation $x = 2$.
This is the equation of the vertical line where the x-coordinate of every point is 2.

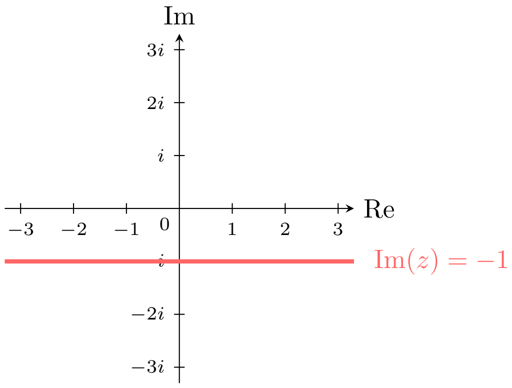


Ex 60: Plot the set of points M in the plane whose affix z satisfies:

$$\mathrm{Im}(z) = -1$$

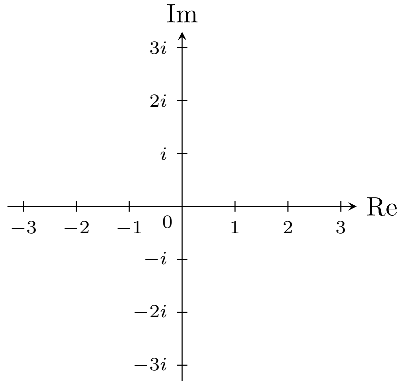


Answer: Let $z = x + yi$. The imaginary part of z is $\mathrm{Im}(z) = y$.
The condition $\mathrm{Im}(z) = -1$ is therefore equivalent to the equation $y = -1$.
This is the equation of the horizontal line where the y-coordinate of every point is -1.

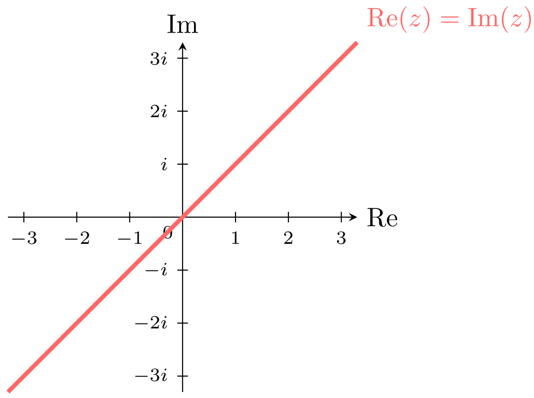


Ex 61: Plot the set of points M in the plane whose affix z satisfies:

$$\mathrm{Re}(z) = \mathrm{Im}(z)$$

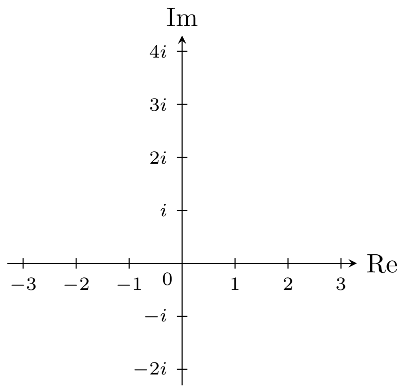


Answer: Let $z = x + yi$. The real part is $\mathrm{Re}(z) = x$ and the imaginary part is $\mathrm{Im}(z) = y$.
The condition $\mathrm{Re}(z) = \mathrm{Im}(z)$ is therefore equivalent to the equation $y = x$.
This is the equation of the line passing through the origin with a slope of 1, also known as the first bisector of the axes.

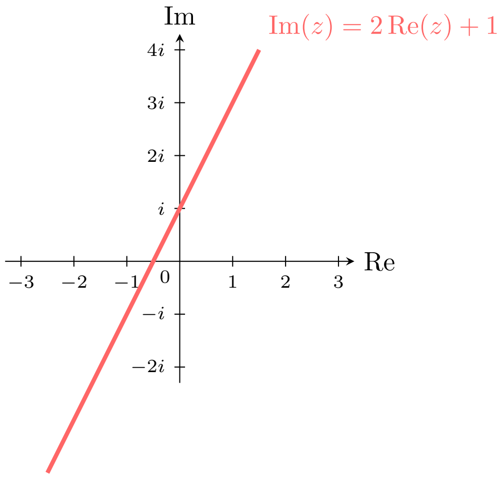


Ex 62: Plot the set of points M in the plane whose affix z satisfies:

$$\mathrm{Im}(z) = 2\,\mathrm{Re}(z) + 1$$



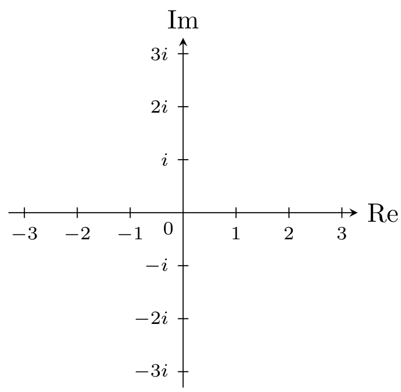
Answer: Let $z = x + yi$. The real part is $\mathrm{Re}(z) = x$ and the imaginary part is $\mathrm{Im}(z) = y$.
The condition $\mathrm{Im}(z) = 2\,\mathrm{Re}(z) + 1$ is therefore equivalent to the equation $y = 2x + 1$.
This is the equation of a straight line with a slope of 2 and a y-intercept of 1.



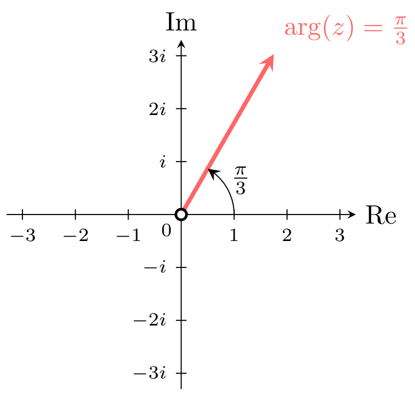
H.2 PLOTTING RAYS IN THE COMPLEX PLANE

Ex 63: Plot the set of points M in the plane whose affix z satisfies:

$$\arg(z) = \frac{\pi}{3}$$

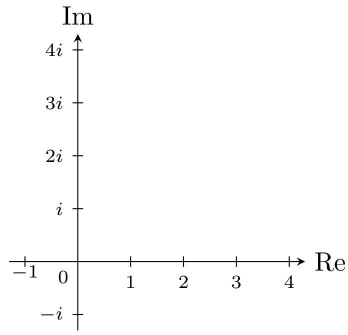


Answer: The condition $\arg(z) = \frac{\pi}{3}$ means that the angle formed by the positive real axis and the vector from the origin to the point $M(z)$ is $\frac{\pi}{3}$ (or 60°).
This set of points forms a ray (a half-line) starting from the origin. The origin itself is excluded because the argument of $z = 0$ is undefined.

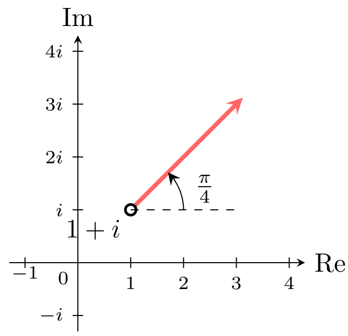


Ex 64: Plot the set of points M in the plane whose affix z satisfies:

$$\arg(z - (1 + i)) = \frac{\pi}{4}$$

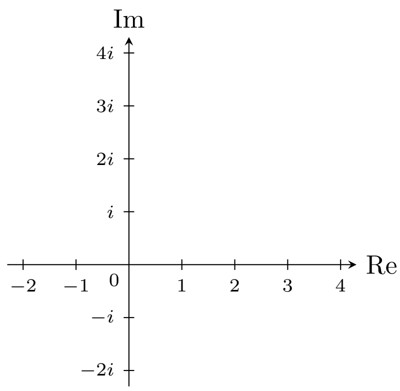


Answer: Let A be the point with affix $z_A = 1 + i$. The condition can be rewritten as $\arg(z - z_A) = \frac{\pi}{4}$.
The complex number $z - z_A$ represents the vector $\overrightarrow{AM}$. Therefore, the condition means that the angle formed by the positive real axis and the vector $\overrightarrow{AM}$ is $\frac{\pi}{4}$ (or 45°).
This set of points forms a ray (a half-line) starting from the point $A(1, 1)$. The point A itself is excluded because if $z = z_A$, the argument is undefined.



Ex 65: Plot the set of points M in the plane whose affix z satisfies:

$$\arg(2z - 2 + 2i) = \frac{\pi}{2}$$



Answer: First, we must factor the expression inside the argument to isolate z.

$$\arg(2z - 2 + 2i) = \arg(2(z - (1 - i)))$$

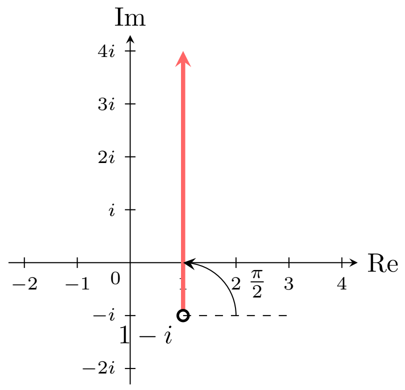
Using the property $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$, we get:

$$\arg(2) + \arg(z - (1 - i)) = \frac{\pi}{2}$$

Since 2 is a positive real number, its argument is $\arg(2) = 0$. The equation simplifies to:

$$\arg(z - (1 - i)) = \frac{\pi}{2}$$

Let A be the point with affix $z_A = 1 - i$. The condition is now $\arg(z - z_A) = \frac{\pi}{2}$.
This means that the angle formed by the positive real axis and the vector $\overrightarrow{AM}$ is $\frac{\pi}{2}$ (or 90°).
This set of points forms a vertical ray starting from the point $A(1, -1)$, pointing upwards. The point A itself is excluded.

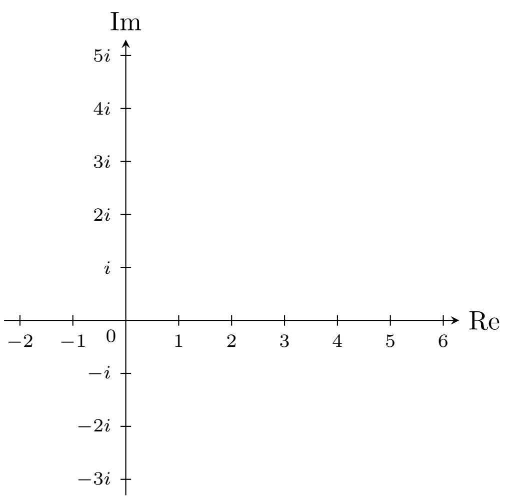


H.3 IDENTIFYING LOCI FROM MODULUS EQUATIONS

Ex 66: Identify the geometric locus of points z in the complex plane that satisfy the equation:

$$|z - (2 + i)| = 3$$

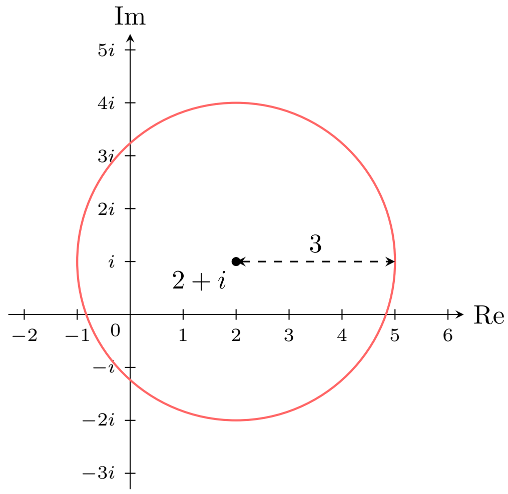
Describe the center and radius of the locus and plot it.



Answer: The equation is in the form $|z - c| = r$.
By comparing the given equation $|z - (2 + i)| = 3$ with the standard form, we can identify:

- The center of the circle is the point with affix $c = 2 + i$.
- The radius of the circle is $r = 3$.

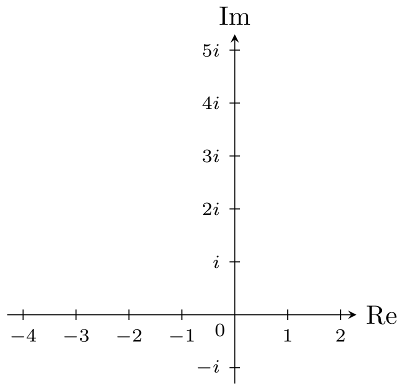
The locus is therefore a circle centered at the point $(2, 1)$ with a radius of 3 units.



Ex 67: Identify the geometric locus of points z in the complex plane that satisfy the equation:

$$|z + 1 - 2i| = 2$$

Describe the center and radius of the locus and plot it.



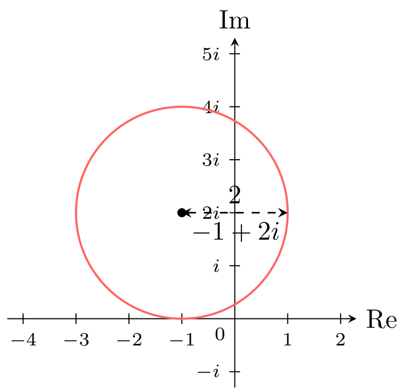
Answer: To match the standard form $|z - c| = r$, we must rewrite the expression inside the modulus.

$$|z + 1 - 2i| = |z - (-1 + 2i)|$$

Now, by comparing $|z - (-1 + 2i)| = 2$ with the standard form, we can identify:

- The center of the circle is the point with affix $c = -1 + 2i$.
- The radius of the circle is $r = 2$.

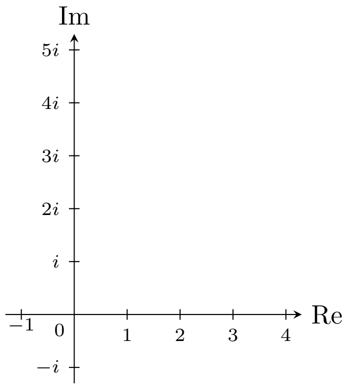
The locus is therefore a circle centered at the point $(-1, 2)$ with a radius of 2 units.



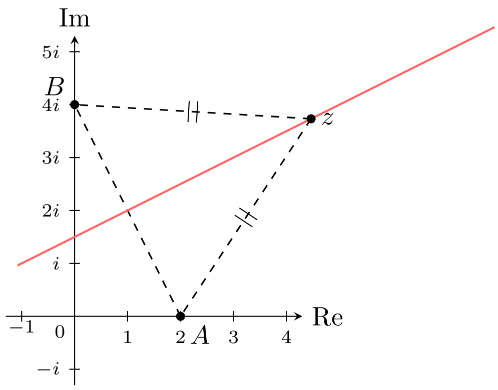
Ex 68: Identify the geometric locus of points z in the complex plane that satisfy the equation:

$$|z - 2| = |z - 4i|$$

Describe the locus and plot it.



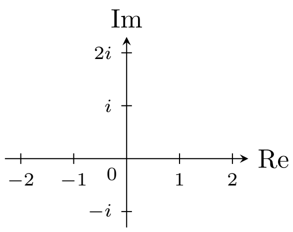
Answer: The equation is in the form $|z-a| = |z-b|$, where $a = 2$ and $b = 4i$.
Let A be the point with affix $z_A = 2$ and B be the point with affix $z_B = 4i$.
The expression $|z-2|$ represents the distance between the point z and the point A.
The expression $|z-4i|$ represents the distance between the point z and the point B.
The equation $|z-2| = |z-4i|$ therefore describes the set of all points z that are equidistant from points A and B.
This is the definition of the **perpendicular bisector** of the line segment [AB].



Ex 69: Identify the geometric locus of points z in the complex plane that satisfy the equation:

$$|z-i| = |z+1|$$

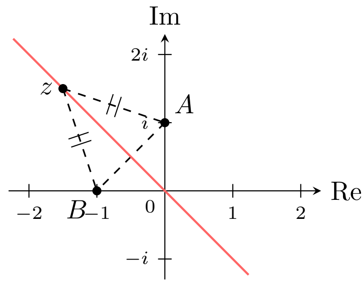
Describe the locus and plot it.



Answer: To match the standard form $|z-a| = |z-b|$, we rewrite the right-hand side of the equation:

$$|z+1| = |z-(-1)|$$

The equation becomes $|z-i| = |z-(-1)|$. This is in the form $|z-a| = |z-b|$, where $a = i$ and $b = -1$.
Let A be the point with affix $z_A = i$ and B be the point with affix $z_B = -1$.
The equation describes the set of all points z that are equidistant from points A and B.
This is the definition of the **perpendicular bisector** of the line segment [AB].



I ROOTS OF COMPLEX NUMBERS

I.1 FINDING THE N-TH ROOTS OF A COMPLEX NUMBER

Ex 70: Find the four 4th roots of unity by solving the equation $z^4 = 1$.

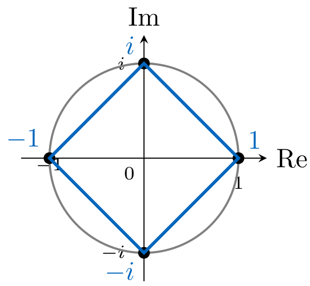
Answer:

$$\begin{aligned} z^4 &= 1 \\ z^4 &= e^{i2k\pi} (k \in \mathbb{Z}) \\ z &= \left(e^{i2k\pi}\right)^{\frac{1}{4}} \\ &= e^{i\frac{2k\pi}{4}} \\ &= e^{i\frac{k\pi}{2}} \end{aligned}$$

To find the four distinct roots, we let k take four consecutive integer values, for example $k = 0, 1, 2, 3$.

- For $k = 0$: $z_0 = e^{i(0)} = 1$
- For $k = 1$: $z_1 = e^{i\frac{\pi}{2}} = i$
- For $k = 2$: $z_2 = e^{i\frac{2\pi}{2}} = e^{i\pi} = -1$
- For $k = 3$: $z_3 = e^{i\frac{3\pi}{2}} = e^{-i\frac{\pi}{2}} = -i$

The four 4th roots of unity are $\{1, i, -1, -i\}$.



Ex 71: Find the three cube roots of unity by solving the equation $z^3 = 1$.

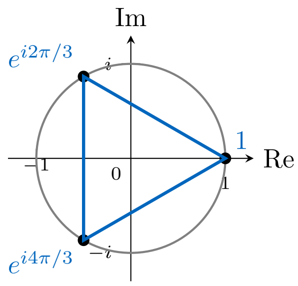
Answer:

$$\begin{aligned} z^3 &= 1 \\ z^3 &= e^{i2k\pi} \quad (k \in \mathbb{Z}) \\ z &= \left(e^{i2k\pi}\right)^{\frac{1}{3}} \\ &= e^{i\frac{2k\pi}{3}} \end{aligned}$$

To find the three distinct roots, we let k take three consecutive integer values, for example $k = 0, 1, 2$.

- For $k = 0$: $z_0 = e^{i(0)} = 1$
- For $k = 1$: $z_1 = e^{i\frac{2\pi}{3}} = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$
- For $k = 2$: $z_2 = e^{i\frac{4\pi}{3}} = \cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3}) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

The three cube roots of unity are $\left\{1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}\right\}$.



Ex 72: Find the three cube roots of $8i$ by solving the equation $z^3 = 8i$.

Answer: First, we write the number $8i$ in its general Euler form. The modulus is $|8i| = 8$. The principal argument is $\arg(8i) = \frac{\pi}{2}$.

$$8i = 8e^{i\left(\frac{\pi}{2}+2k\pi\right)} \quad (k \in \mathbb{Z})$$

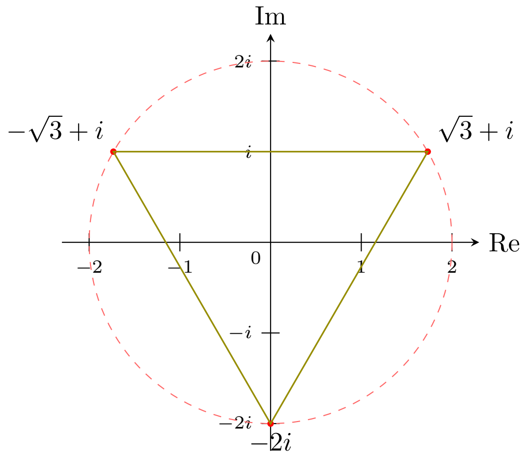
Now we solve for z:

$$\begin{aligned} z^3 &= 8e^{i\left(\frac{\pi}{2}+2k\pi\right)} \\ z &= \left(8e^{i\left(\frac{\pi}{2}+2k\pi\right)}\right)^{\frac{1}{3}} \\ z &= 8^{\frac{1}{3}} e^{i\frac{\frac{\pi}{2}+2k\pi}{3}} \\ z &= 2e^{i\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right)} \end{aligned}$$

To find the three distinct roots, we let $k = 0, 1, 2$.

- For $k = 0$: $z_0 = 2e^{i\frac{\pi}{6}} = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \sqrt{3} + i$
- For $k = 1$: $z_1 = 2e^{i(\frac{\pi}{6}+\frac{2\pi}{3})} = 2e^{i\frac{5\pi}{6}} = -\sqrt{3} + i$
- For $k = 2$: $z_2 = 2e^{i(\frac{\pi}{6}+\frac{4\pi}{3})} = 2e^{i\frac{9\pi}{6}} = 2e^{i\frac{3\pi}{2}} = -2i$

The three cube roots of $8i$ are $\{\sqrt{3} + i, -\sqrt{3} + i, -2i\}$.



Ex 73: Find the four 4th roots of -4 by solving the equation $z^4 = -4$.

Answer: First, we write the number -4 in its general Euler form. The modulus is $|-4| = 4$. The principal argument is $\arg(-4) = \pi$.

$$-4 = 4e^{i(\pi+2k\pi)} \quad (k \in \mathbb{Z})$$

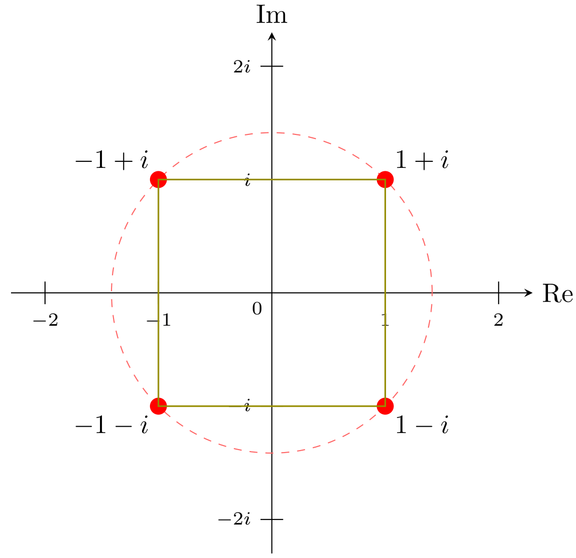
Now we solve for z:

$$\begin{aligned} z^4 &= 4e^{i(\pi+2k\pi)} \\ z &= \left(4e^{i(\pi+2k\pi)}\right)^{\frac{1}{4}} \\ z &= 4^{\frac{1}{4}} e^{i\frac{\pi+2k\pi}{4}} \\ z &= \sqrt{2}e^{i\left(\frac{\pi}{4}+\frac{k\pi}{2}\right)} \quad (4^{1/4} = \left(2^2\right)^{1/4} = 2^{1/2} = \sqrt{2}) \end{aligned}$$

To find the four distinct roots, we let $k = 0, 1, 2, 3$. The modulus of each root is $r = \sqrt{2}$.

- For $k = 0$: $z_0 = \sqrt{2}e^{i\frac{\pi}{4}} = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1 + i$
- For $k = 1$: $z_1 = \sqrt{2}e^{i\frac{3\pi}{4}} = \sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -1 + i$
- For $k = 2$: $z_2 = \sqrt{2}e^{i\frac{5\pi}{4}} = \sqrt{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -1 - i$
- For $k = 3$: $z_3 = \sqrt{2}e^{i\frac{7\pi}{4}} = \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = 1 - i$

The four 4th roots of -4 are $\{1+i, -1+i, -1-i, 1-i\}$, or more compactly $\{1 \pm i, -1 \pm i\}$.



Ex 74: Find the three cube roots of $4 + i4\sqrt{3}$ by solving the equation $z^3 = 4 + i4\sqrt{3}$.

Answer: First, we write the number $c = 4 + i4\sqrt{3}$ in its general Euler form.
The modulus is $|c| = \sqrt{4^2 + (4\sqrt{3})^2} = \sqrt{16 + 48} = \sqrt{64} = 8$.
The principal argument is found by factoring: $c = 8(\frac{1}{2} + i\frac{\sqrt{3}}{2})$. Since $\cos\theta = \frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$, we have $\arg(c) = \frac{\pi}{3}$.

$$c = 8e^{i\left(\frac{\pi}{3}+2k\pi\right)} \quad (k \in \mathbb{Z})$$

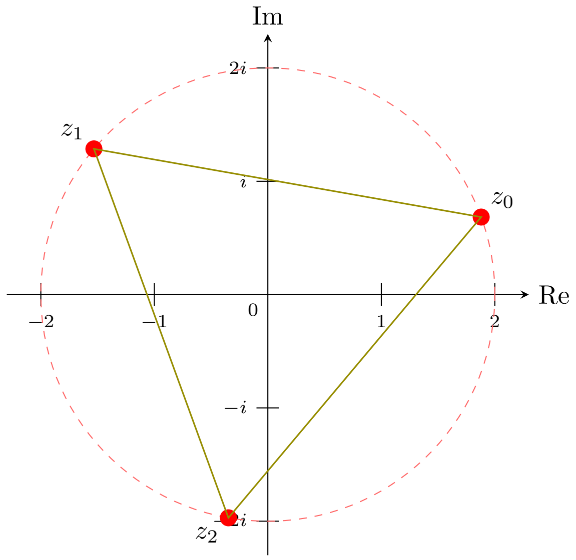
Now we solve for z:

$$\begin{aligned} z^3 &= 8e^{i\left(\frac{\pi}{3}+2k\pi\right)} \\ z &= \left(8e^{i\left(\frac{\pi}{3}+2k\pi\right)}\right)^{\frac{1}{3}} \\ z &= 8^{\frac{1}{3}} e^{i\frac{\frac{\pi}{3}+2k\pi}{3}} \\ z &= 2e^{i\left(\frac{\pi}{9}+\frac{2k\pi}{3}\right)} \end{aligned}$$

To find the three distinct roots, we let $k = 0, 1, 2$. The modulus of each root is $r = 2$.

- For $k = 0$: $z_0 = 2e^{i\frac{\pi}{9}}$
- For $k = 1$: $z_1 = 2e^{i(\frac{\pi}{9}+\frac{2\pi}{3})} = 2e^{i\frac{7\pi}{9}}$
- For $k = 2$: $z_2 = 2e^{i(\frac{\pi}{9}+\frac{4\pi}{3})} = 2e^{i\frac{13\pi}{9}}$

The three cube roots are $\left\{2e^{i\frac{\pi}{9}}, 2e^{i\frac{7\pi}{9}}, 2e^{i\frac{13\pi}{9}}\right\}$.


Im
2i
i
−i
−2i
Re
−2
−1
0
1
2
z_0
z_1
z_2