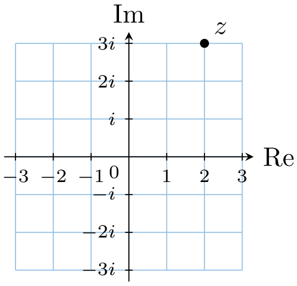
COMPLEX NUMBERS: GEOMETRICAL APPROACH

A COMPLEX PLANE

A.1 READING THE AFFIX OF A POINT

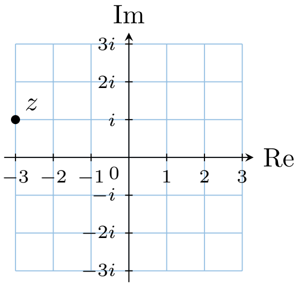
Ex 1:



Find the components of z:

$z =$

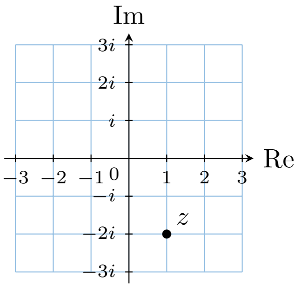
Ex 2:



Find the components of z:

$z =$

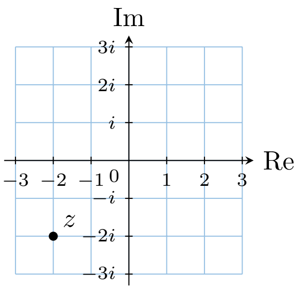
Ex 3:



Find the components of z:

$z =$

Ex 4:

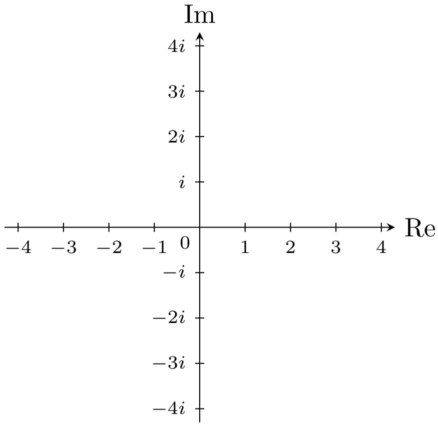


Find the components of z:

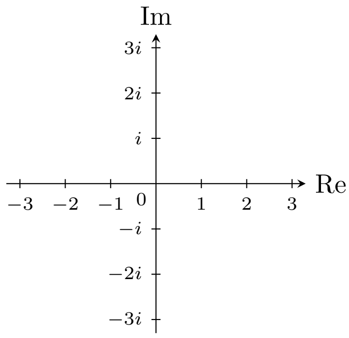
$z =$

A.2 CONJECTURING THE NATURE OF A FIGURE

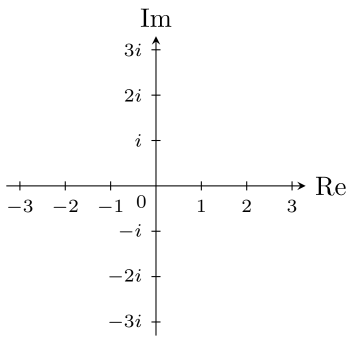
Ex 5: Plot the points A, B, C, and D with respective affixes $z_A = 2 - 2i$, $z_B = 1$, $z_C = 3 + i$, and $z_D = 4 - i$.
Conjecture the nature of the quadrilateral $ABCD$.



Ex 6: Plot the points A, B, and C with respective affixes $z_A = -2 - i$, $z_B = 2 - i$, and $z_C = i$.
Conjecture the nature of the triangle ABC.



Ex 7: Plot the points A, B, and I with respective affixes $z_A = -3 + 2i$, $z_B = 1 - 2i$, and $z_I = -1$.
Conjecture the geometric relationship between the points A, B, and I.



B MODULUS AND ARGUMENT

B.1 CALCULATING THE MODULUS OF A COMPLEX NUMBER

Ex 8: Calculate the modulus of the complex number $z = 3 + 4i$.

$|z| = \square$

Ex 9: Calculate the modulus of the complex number $z = -1 + 2i$.

$|z| = \square$

Ex 10: Calculate the modulus of the complex number $z = -5i$.

$|z| = \square$

Ex 11: Calculate the modulus of the complex number $z = 1 - i$.

$|z| = \square$

B.2 CALCULATING THE ARGUMENT OF A COMPLEX NUMBER

Ex 12: Find the principal argument of the complex number $z = -1 + i\sqrt{3}$. (i.e., the argument in the interval $(-\pi, \pi]$).

$\arg(z) = \square$

Ex 13: Find the principal argument of the complex number $z = 1 - i$. (i.e., the argument in the interval $(-\pi, \pi]$).

$\arg(z) = \square$

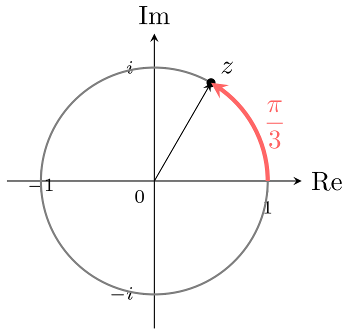
Ex 14: Find the principal argument of the complex number $z = \frac{3\sqrt{3}}{2} + i\frac{3}{2}$. (i.e., the argument in the interval $(-\pi, \pi]$).

$\arg(z) = \square$

C UNIT MODULUS COMPLEX NUMBERS AND THE IMAGINARY EXPONENTIAL

C.1 FINDING THE AFFIX OF A POINT ON THE UNIT CIRCLE

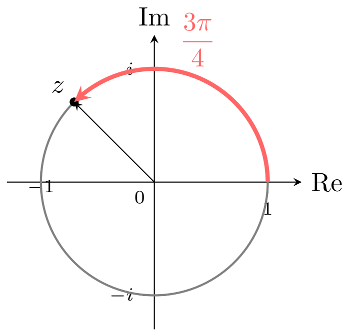
Ex 15:



Find the standard form of the affix z shown in the diagram.

$z = \square$

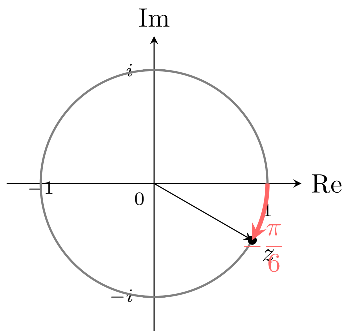
Ex 16:



Find the standard form of the affix z shown in the diagram.

$z = \square$

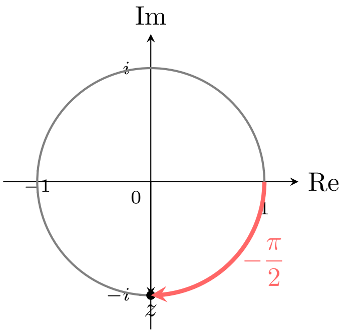
Ex 17:



Find the standard form of the affix z shown in the diagram.

$z = \square$

Ex 18:



Find the standard form of the affix z shown in the diagram.

$$z = \square$$

C.2 EVALUATING COMPLEX EXPONENTIALS

Ex 19: Convert the following complex number to standard form:

$$z = e^{i\frac{\pi}{3}}$$

$$z = \square$$

Ex 20: Convert the following complex number to standard form:

$$z = e^{-i\frac{\pi}{2}}$$

$$z = \square$$

Ex 21: Convert the following complex number to standard form:

$$z = e^{i\frac{4\pi}{3}}$$

$$z = \square$$

Ex 22: Convert the following complex number to standard form:

$$z = e^{i\frac{13\pi}{6}}$$

$$z = \square$$

C.3 APPLYING THE PROPERTIES OF EXPONENTS

Ex 23: Given $z_1 = e^{i\frac{2\pi}{3}}$ and $z_2 = e^{i\frac{\pi}{3}}$, calculate and simplify the product z_1z_2.

$$z_1z_2 = \square$$

Ex 24: Given $z_1 = e^{i\frac{\pi}{2}}$ and $z_2 = e^{i\frac{\pi}{6}}$, calculate and simplify the quotient $\frac{z_1}{z_2}$.

$$\frac{z_1}{z_2} = \square$$

Ex 25: Given $z_1 = e^{i\frac{3\pi}{4}}$ and $z_2 = e^{-i\frac{\pi}{2}}$, calculate and simplify the product z_1z_2.

$$z_1z_2 = \square$$

Ex 26: Given $z = e^{i\frac{2\pi}{3}}$, calculate z^3.

$$z^3 = \square$$

Ex 27: Given $z = e^{i\frac{\pi}{4}}$, find its conjugate $\overline{z}$. Give your answer in Euler's form with a principal argument.

$$\overline{z} = \square$$

Ex 28: Given $z = e^{i\frac{2\pi}{3}}$, find its conjugate $\overline{z}$. Give your answer in Euler's form with a principal argument.

$$\overline{z} = \square$$

D POLAR AND EULER'S FORMS

D.1 CONVERTING FROM POLAR TO STANDARD FORM

Ex 29: Convert the following complex number from polar form to standard form:

$$z_1 = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$$

$$z = \square$$

Ex 30: Convert the following complex number from polar form to standard form:

$$z = 3\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$

$$z = \square$$

Ex 31: Convert the following complex number from polar form to standard form:

$$z = 4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

$$z = \square$$

Ex 32: Convert the following complex number from polar form to standard form:

$$z = \frac{1}{2}\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right)$$

$$z = \square$$

D.2 CONVERTING FROM STANDARD TO POLAR FORM

Ex 33: Convert the complex number $z = -1 + i\sqrt{3}$ to polar form.

$$z = \square$$

Ex 34: Convert the complex number $z = 1 - i$ to polar form.

$$z = \square$$

Ex 35: Convert the complex number $z = \frac{3\sqrt{3}}{2} + i\frac{3}{2}$ to polar form.

$$z = \square$$

D.3 CONVERTING FROM POLAR TO EULER'S FORM

Ex 36: Convert the following complex number from polar form to Euler's form:

$$z = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$$

$$z = \boxed{}$$

Ex 37: Convert the following complex number from polar form to Euler's form:

$$z = 5\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

$$z = \boxed{}$$

Ex 38: Convert the following complex number from polar form to Euler's form:

$$z = \sqrt{3}\left(\cos(\pi) + i\sin(\pi)\right)$$

$$z = \boxed{}$$

E DE MOIVRE'S THEOREM

E.1 APPLYING DE MOIVRE'S THEOREM

Ex 39: Write $(1+i)^8$ in standard form.

$$(1+i)^8 = \boxed{}$$

Ex 40: Write $(\sqrt{3}-i)^6$ in standard form.

$$(\sqrt{3}-i)^6 = \boxed{}$$

Ex 41: Write $(-2-2i)^3$ in standard form.

$$(-2-2i)^3 = \boxed{}$$

F PROPERTIES OF MODULUS AND ARGUMENT

F.1 PROVING THE PROPERTIES OF THE MODULUS

Ex 42: Prove that $|\overline{z}| = |z|$ for any complex number z.

Ex 43: Prove that $|z|^2 = z\overline{z}$ for any complex number z.

Ex 44: Prove that $|z_1 z_2| = |z_1||z_2|$ for any complex numbers z_1 and z_2.

F.2 PROVING THE PROPERTIES OF THE ARGUMENT

Ex 45: Prove that $\arg(\overline{z}) = -\arg(z) \pmod{2\pi}$ for any non-zero complex number z.

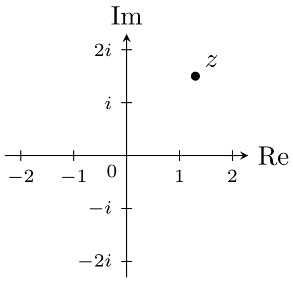
Ex 46: Prove that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi}$ for any non-zero complex numbers z_1 and z_2.

Ex 47: Prove that $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \pmod{2\pi}$ for any non-zero complex numbers z_1 and z_2.

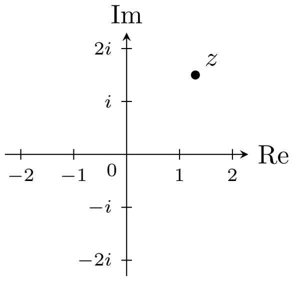
G GEOMETRY IN THE COORDINATE PLANE

G.1 VISUALIZING FUNDAMENTAL TRANSFORMATIONS

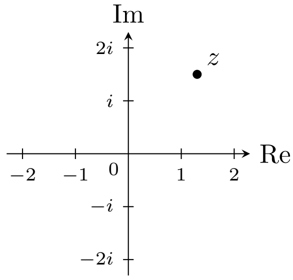
Ex 48: Given the point with affix z below, plot the point with affix $\overline{z}$.



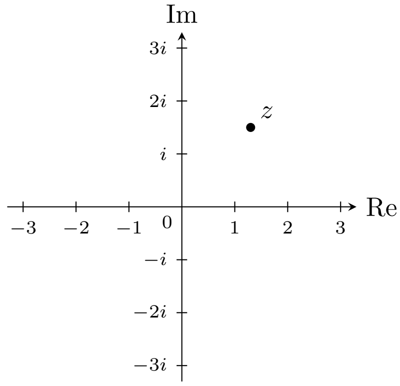
Ex 49: Given the point with affix z below, plot the point with affix iz.



Ex 50: Given the point with affix z below, plot the point with affix $-z$.



Ex 51: Given the point with affix z below, plot the point with affix $2z$.



G.2 CALCULATING DISTANCES, MIDPOINTS, AND ANGLES

Ex 52: Given the points $A(2,3)$ and $B(6,1)$ on the Cartesian plane. Use complex numbers to find:

1. the distance AB
2. the midpoint of the segment $[AB]$.

Ex 53: Given the points $A(-1,5)$ and $B(3,-1)$ on the Cartesian plane. Use complex numbers to find:

1. the distance AB
2. the midpoint of the segment $[AB]$.

Ex 54: Let A, B, and C be three points in the complex plane with respective affixes $z_A = 1$, $z_B = 3$, and $z_C = 3 + 2i\sqrt{3}$. Calculate the measure of the angle $\angle BAC$.

$$\angle BAC = \square$$

Ex 55: Let A, B, and C be three points in the complex plane with respective affixes $z_A = 1+i$, $z_B = -1+3i$, and $z_C = 2+2i$. Calculate the measure of the angle $\angle BAC$.

$$\angle BAC = \square$$

G.3 PROVING THE NATURE OF GEOMETRIC FIGURES

Ex 56: Let A, B, and C be three points in the complex plane with respective affixes $z_A = 1 + 2i$, $z_B = 3 + 3i$, and $z_C = 2$. Prove that the triangle ABC is isosceles at A.

Ex 57: Let A, B, and C be three points in the complex plane with respective affixes $z_A = 1 + i$, $z_B = 3 + 2i$, and $z_C = 2 + 4i$. Prove that the triangle ABC is a right-angled isosceles triangle at B.

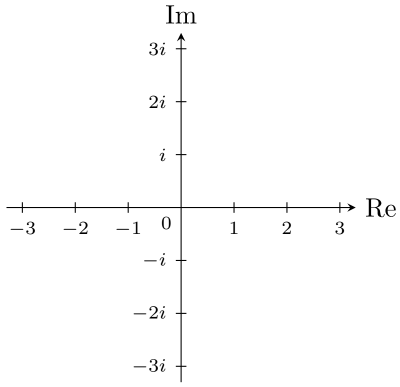
Ex 58: Let A, B, C, and D be four points in the complex plane with respective affixes $z_A = 1$, $z_B = 3 + i$, $z_C = 2 + 3i$, and $z_D = 2i$.
Prove that the quadrilateral ABCD is a square.

H GEOMETRIC LOCI IN THE COMPLEX PLANE

H.1 PLOTTING LINES

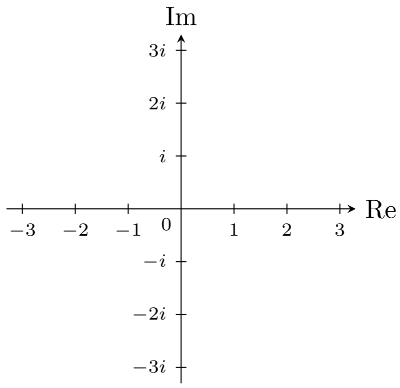
Ex 59: Plot the set of points M in the plane whose affix z satisfies:

$$\mathrm{Re}(z) = 2$$



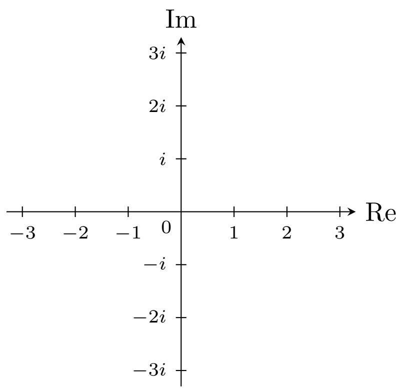
Ex 60: Plot the set of points M in the plane whose affix z satisfies:

$$\mathrm{Im}(z) = -1$$



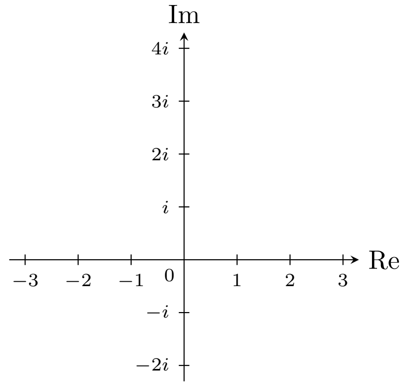
Ex 61: Plot the set of points M in the plane whose affix z satisfies:

$$\mathrm{Re}(z) = \mathrm{Im}(z)$$



Ex 62: Plot the set of points M in the plane whose affix z satisfies:

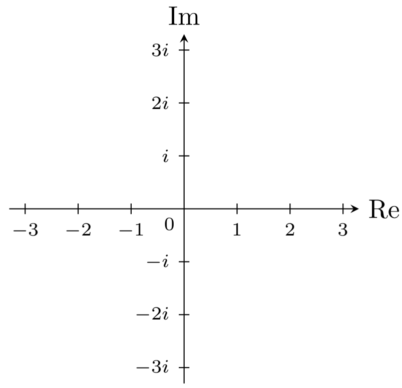
$$\mathrm{Im}(z) = 2\,\mathrm{Re}(z) + 1$$



H.2 PLOTTING RAYS IN THE COMPLEX PLANE

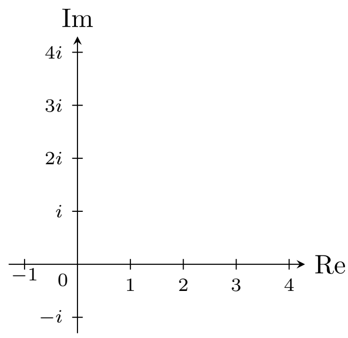
Ex 63: Plot the set of points M in the plane whose affix z satisfies:

$$\arg(z) = \frac{\pi}{3}$$



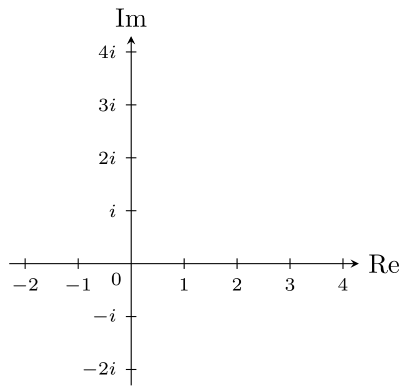
Ex 64: Plot the set of points M in the plane whose affix z satisfies:

$$\arg(z - (1 + i)) = \frac{\pi}{4}$$



Ex 65: Plot the set of points M in the plane whose affix z satisfies:

$$\arg(2z - 2 + 2i) = \frac{\pi}{2}$$

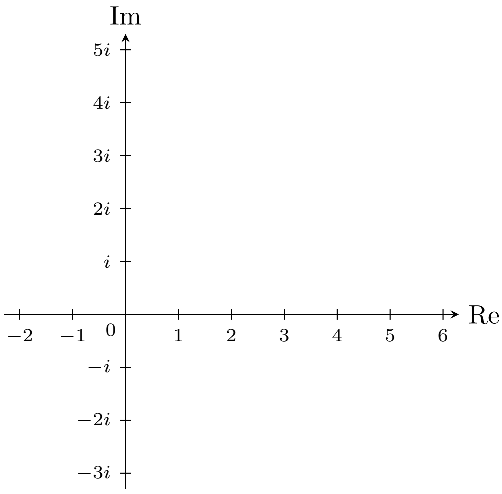


H.3 IDENTIFYING LOCI FROM MODULUS EQUATIONS

Ex 66: Identify the geometric locus of points z in the complex plane that satisfy the equation:

$$|z-(2+i)|=3$$

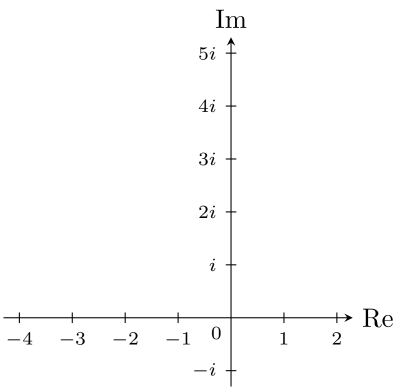
Describe the center and radius of the locus and plot it.



Ex 67: Identify the geometric locus of points z in the complex plane that satisfy the equation:

$$|z+1-2i|=2$$

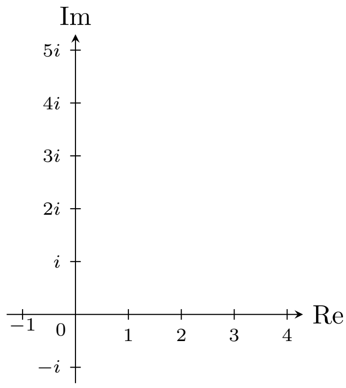
Describe the center and radius of the locus and plot it.



Ex 68: Identify the geometric locus of points z in the complex plane that satisfy the equation:

$$|z-2|=|z-4i|$$

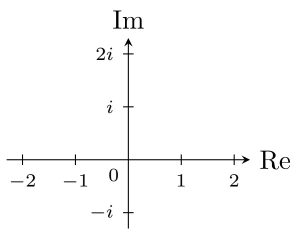
Describe the locus and plot it.



Ex 69: Identify the geometric locus of points z in the complex plane that satisfy the equation:

$$|z-i|=|z+1|$$

Describe the locus and plot it.



I ROOTS OF COMPLEX NUMBERS

I.1 FINDING THE N-TH ROOTS OF A COMPLEX NUMBER

Ex 70: Find the four 4th roots of unity by solving the equation $z^4=1$.

Ex 71: Find the three cube roots of unity by solving the equation $z^3 = 1$.

Ex 72: Find the three cube roots of $8i$ by solving the equation $z^3 = 8i$.

Ex 73: Find the four 4th roots of -4 by solving the equation $z^4 = -4$.

Ex 74: Find the three cube roots of $4 + i4\sqrt{3}$ by solving the equation $z^3 = 4 + i4\sqrt{3}$.