COMPLEX NUMBERS: ALGEBRAIC APPROACH

The system of real numbers, while vast, is incomplete. A simple quadratic equation such as $x^2 = -1$ has no solution within the real numbers. To address this, we extend the real number line into a two-dimensional plane, introducing a new number, the imaginary unit i. This extension forms the set of complex numbers, $\mathbb{C}$, a system where not only does $x^2 = -1$ have a solution, but every non-constant polynomial with real (or complex) coefficients has a complete set of solutions in $\mathbb{C}$ (Fundamental Theorem of Algebra). This chapter introduces the algebraic foundations of complex numbers, their operations, and their power in solving equations.

A THE NUMBER i AND THE SET OF COMPLEX NUMBERS

Definition The Imaginary Unit i

The **imaginary unit**, denoted by i, is defined as a number such that:

$$i^2 = -1$$

Definition Complex Number

A **complex number** is a number of the form $z = a + bi$, where a and b are real numbers and i is the imaginary unit. The set of all complex numbers is denoted by $\mathbb{C}$.

Note

- i cannot be a real number, since no real number squared is equal to -1.
- A real number a is a complex number where $b = 0$, written as $a = a + 0i$. Thus $\mathbb{R} \subset \mathbb{C}$.
- A number where $a = 0$, such as bi, is called a **purely imaginary number**.

Definition Real and Imaginary Parts

For a complex number $z = a + bi$ (with $a, b \in \mathbb{R}$):

- a is the **real part** of z, denoted $\text{Re}(z)$.
- b is the **imaginary part** of z, denoted $\text{Im}(z)$.

By definition, $\text{Re}(z)$ and $\text{Im}(z)$ are real numbers.

Ex: For $z = 2 + 3i$:

- $\text{Re}(2 + 3i) = 2$
- $\text{Im}(2 + 3i) = 3$

B OPERATIONS WITH COMPLEX NUMBERS

The arithmetic of complex numbers follows the standard rules of algebra, together with the key relation $i^2 = -1$.

Definition **Algebraic Operations**

Let $z = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$.

- **Addition**:
$$\begin{aligned} z + w &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i. \end{aligned}$$

- **Subtraction**:
$$\begin{aligned} z - w &= (a + bi) - (c + di) \\ &= (a - c) + (b - d)i. \end{aligned}$$

- **Multiplication**:
$$\begin{aligned} z \times w &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \quad (i^2 = -1) \\ &= (ac - bd) + (ad + bc)i. \end{aligned}$$

- **Division** (with $w \neq 0$):
$$\begin{aligned} \frac{z}{w} = \frac{a + bi}{c + di} &= \left(\frac{a + bi}{c + di}\right)\left(\frac{c - di}{c - di}\right) \\ &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{ac - adi + bci - bdi^2}{c^2 - (di)^2} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}. \end{aligned}$$

C EQUALITY OF COMPLEX NUMBERS

Proposition **Equality of Complex Numbers**

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal:

$$a + bi = c + di \Leftrightarrow a = c \text{ and } b = d.$$

Proof

- ($\Rightarrow$) Suppose $a + bi = c + di$. Then
$$(a - c) + (b - d)i = 0.$$
This implies $a - c = i(d - b)$. If $d \neq b$, then
$$i = \frac{a - c}{d - b}.$$
Since a, b, c, d are real, the right-hand side is a real number, while i is not real (because $i^2 = -1$ has no real solution). This is a contradiction. Therefore, the assumption $d \neq b$ must be false, which means $d = b$. Substituting this back into $a - c = i(d - b)$ gives $a - c = 0$, so $a = c$.

- ($\Leftarrow$) Suppose $a = c$ and $b = d$. Then $bi = di$. Adding the equalities $a = c$ and $bi = di$ gives
$$a + bi = c + di.$$

D COMPLEX CONJUGATE

Definition **Complex Conjugate**

The **complex conjugate** of $z = a + bi$ (with $a, b \in \mathbb{R}$) is $\overline{z} = a - bi$. The complex conjugate of z is often denoted as $\overline{z}$ or z^*.

Ex: The complex conjugate of $2 + 3i$ is $\overline{2 + 3i} = 2 - 3i$.

Proposition Properties of the Conjugate

Given two complex numbers z and w:

$$\begin{aligned} \overline{\overline{z}} &= z, \\ \overline{z+w} &= \overline{z} + \overline{w}, \\ \overline{z-w} &= \overline{z} - \overline{w}, \\ \overline{zw} &= \overline{z}\,\overline{w}, \\ \overline{\left(\frac{z}{w}\right)} &= \frac{\overline{z}}{\overline{w}}, \quad \text{if } w \neq 0. \end{aligned}$$

Proof

Let $z = a + bi$ and $w = c + di$.For example,

$$\begin{aligned} \overline{z+w} &= \overline{(a+c) + (b+d)i} \\ &= (a+c) - (b+d)i \\ &= a - bi + c - di \\ &= \overline{z} + \overline{w}. \end{aligned}$$

The other properties are proven similarly, by expanding and using $i^2 = -1$.