COMPLEX NUMBERS: ALGEBRAIC APPROACH

A THE NUMBER i AND THE SET OF COMPLEX NUMBERS

A.1 IDENTIFYING REAL AND IMAGINARY PARTS

Ex 1: Find the real part of the following complex number:

$$\mathrm{Re}(2+3i) = \boxed{2}$$

Answer: For a complex number written in the standard form $z = a + bi$, the real part is the component 'a' that is not multiplied by i.

$$z = 2 + 3i$$

Thus, the real part is $\mathrm{Re}(2+3i) = 2$.

Ex 2: Find the imaginary part of the following complex number:

$$\mathrm{Im}(2-3i) = \boxed{-3}$$

Answer: For a complex number written in the standard form $z = a + bi$, the imaginary part is the component 'b' that is multiplied by i.

$$z = 2 + (-3)i$$

Thus, the imaginary part is $\mathrm{Im}(2-3i) = -3$.

Ex 3: Find the real part of the following complex number:

$$\mathrm{Re}(3i) = \boxed{0}$$

Answer: For a complex number written in the standard form $z = a + bi$, the real part is the component 'a'. A purely imaginary number like $3i$ can be written with a zero in the real part.

$$z = 0 + 3i$$

Thus, the real part is $\mathrm{Re}(0+3i) = 0$.

Ex 4: Find the imaginary part of the following complex number:

$$\mathrm{Im}(2) = \boxed{0}$$

Answer: For a complex number written in the standard form $z = a + bi$, the imaginary part is the component 'b'. A real number like 2 can be written with a zero in the imaginary part.

$$z = 2 + 0i$$

Thus, the imaginary part is $\mathrm{Im}(2+0i) = 0$.

Ex 5: Find the real part of the following complex number:

$$\mathrm{Re}(1+\sqrt{2}+3i) = \boxed{1+\sqrt{2}}$$

Answer: For a complex number written in the standard form $z = a + bi$, the real part is the component 'a'. In this case, the real part consists of all terms that are not multiplied by i.

$$z = (1+\sqrt{2}) + 3i$$

Thus, the real part is $\mathrm{Re}(1+\sqrt{2}+3i) = 1+\sqrt{2}$.

A.2 CLASSIFYING COMPLEX NUMBERS

MCQ 6: $1+2i$ is a purely imaginary number.

☐ True

☒ False

Answer: A purely imaginary number is a complex number $z = a + bi$ where the real part a is zero.
For the complex number $z = 1 + 2i$, the real part is $\mathrm{Re}(z) = 1$.
Since the real part is not zero, the number is not purely imaginary.

MCQ 7: $\sqrt{2}i$ is a purely imaginary number.

☒ True

☐ False

Answer: A purely imaginary number is a complex number $z = a + bi$ where the real part a is zero.
The complex number $z = \sqrt{2}i$ can be written as $z = 0 + \sqrt{2}i$.
The real part is $\mathrm{Re}(z) = 0$.
Since the real part is zero, the number is purely imaginary.

MCQ 8: $1+2i$ is a real number.

☐ True

☒ False

Answer: A real number is a complex number $z = a + bi$ where the imaginary part b is zero.
For the complex number $z = 1+2i$, the imaginary part is $\mathrm{Im}(z) = 2$.
Since the imaginary part is not zero, the number is not a real number.

MCQ 9: $\sqrt{2}$ is a real number.

☒ True

☐ False

Answer: A real number is a complex number $z = a + bi$ where the imaginary part b is zero.
The number $z = \sqrt{2}$ can be written as $z = \sqrt{2} + 0i$. The imaginary part is $\mathrm{Im}(z) = 0$.
Since the imaginary part is zero, the number is a real number.

B OPERATIONS WITH COMPLEX NUMBERS

B.1 CALCULATING WITH COMPLEX NUMBERS

Ex 10: For $z = 2+3i$ and $w = 4-5i$, write in standard form:

$$z + w = \boxed{6-2i}$$

Answer:

$$\begin{aligned} z + w &= (2+3i) + (4-5i) \\ &= (2+4) + (3i - 5i) \\ &= 6 - 2i \end{aligned}$$

Ex 11: For $z = 2-4i$ and $w = -2+3i$, write in standard form:

$$z - w = \boxed{4 - 7i}$$

Answer:

$$\begin{aligned} z - w &= (2 - 4i) - (-2 + 3i) \\ &= 2 - 4i + 2 - 3i \\ &= (2 + 2) + (-4i - 3i) \\ &= 4 - 7i \end{aligned}$$

Ex 12: For $z = i$ and $w = 2 + i$, write in standard form:

$$zw = \boxed{-1 + 2i}$$

Answer:

$$\begin{aligned} zw &= i(2 + i) \\ &= i \cdot 2 + i \cdot i \\ &= 2i + i^2 \\ &= 2i - 1 \\ &= -1 + 2i \end{aligned}$$

Ex 13: For $z = 2 - i$ and $w = 1 + 3i$, write in standard form:

$$zw = \boxed{5 + 5i}$$

Answer:

$$\begin{aligned} zw &= (2 - i)(1 + 3i) \\ &= 2(1) + 2(3i) - i(1) - i(3i) \\ &= 2 + 6i - i - 3i^2 \\ &= 2 + 5i - 3(-1) \\ &= 2 + 5i + 3 \\ &= 5 + 5i \end{aligned}$$

B.2 DIVIDING COMPLEX NUMBERS

Ex 14: For $z = 2$ and $w = 1 + i$, write in standard form:

$$\frac{z}{w} = \boxed{1 - i}$$

Answer:

$$\begin{aligned} \frac{z}{w} &= \frac{2}{1 + i} \\ &= \frac{2}{1 + i} \times \frac{1 - i}{1 - i} \\ &= \frac{2(1 - i)}{(1 + i)(1 - i)} \\ &= \frac{2 - 2i}{1^2 - i^2} \\ &= \frac{2 - 2i}{1 - (-1)} \\ &= \frac{2 - 2i}{2} \\ &= 1 - i \end{aligned}$$

Ex 15: For $z = 2i$ and $w = i - 1$, write in standard form:

$$\frac{z}{w} = \boxed{1 - i}$$

Answer: First, we write w in standard form as $w = -1 + i$. The conjugate of the denominator is $\overline{w} = -1 - i$.

$$\begin{aligned} \frac{z}{w} &= \frac{2i}{-1 + i} \\ &= \frac{2i}{-1 + i} \times \frac{-1 - i}{-1 - i} \\ &= \frac{2i(-1 - i)}{(-1 + i)(-1 - i)} \\ &= \frac{-2i - 2i^2}{(-1)^2 - i^2} \\ &= \frac{-2i - 2(-1)}{1 - (-1)} \\ &= \frac{2 - 2i}{2} \\ &= 1 - i \end{aligned}$$

Ex 16: For $z = 2 - i$ and $w = i - 1$, write in standard form:

$$\frac{z}{w} = \boxed{-3/2 - 1/2i}$$

Answer: First, we write w in standard form as $w = -1 + i$. The conjugate of the denominator is $\overline{w} = -1 - i$.

$$\begin{aligned} \frac{z}{w} &= \frac{2 - i}{-1 + i} \\ &= \frac{2 - i}{-1 + i} \times \frac{-1 - i}{-1 - i} \\ &= \frac{(2 - i)(-1 - i)}{(-1 + i)(-1 - i)} \\ &= \frac{2(-1) + 2(-i) - i(-1) - i(-i)}{(-1)^2 - i^2} \\ &= \frac{-2 - 2i + i + i^2}{1 - (-1)} \\ &= \frac{-2 - i - 1}{2} \\ &= \frac{-3 - i}{2} \\ &= -\frac{3}{2} - \frac{1}{2}i \end{aligned}$$

Ex 17: For $z = i$ and $w = 2 - i$, write in standard form:

$$\frac{z}{w} = \boxed{-1/5 + 2/5i}$$

Answer:

$$\begin{aligned} \frac{z}{w} &= \frac{i}{2 - i} \\ &= \frac{i}{2 - i} \times \frac{2 + i}{2 + i} \\ &= \frac{i(2 + i)}{(2 - i)(2 + i)} \\ &= \frac{2i + i^2}{2^2 - i^2} \\ &= \frac{2i - 1}{4 - (-1)} \\ &= \frac{-1 + 2i}{5} \\ &= -\frac{1}{5} + \frac{2}{5}i \end{aligned}$$

B.3 SIMPLIFYING EXPRESSIONS TO FIND REAL AND IMAGINARY PARTS

Ex 18: Find the real part of the following complex number:

$$\mathrm{Re}(i(2+i)) = \boxed{-1}$$

Answer: First, we must simplify the expression inside the parentheses to write it in the standard form $z = a + bi$.

$$\begin{aligned} i(2+i) &= i \cdot 2 + i \cdot i \\ &= 2i + i^2 \\ &= 2i - 1 \\ &= -1 + 2i \end{aligned}$$

For a complex number in the form $z = a+bi$, the real part is 'a'.
Thus, the real part is $\mathrm{Re}(-1+2i) = -1$.

Ex 19: Find the imaginary part of the following complex number:

$$\mathrm{Im}((1+i)(3-2i)) = \boxed{1}$$

Answer: First, we must simplify the expression by multiplication to write it in the standard form $z = a + bi$.

$$\begin{aligned} (1+i)(3-2i) &= 1(3) + 1(-2i) + i(3) + i(-2i) \\ &= 3 - 2i + 3i - 2i^2 \\ &= 3 + i - 2(-1) \\ &= 3 + i + 2 \\ &= 5 + 1i \end{aligned}$$

For a complex number in the form $z = a+bi$, the imaginary part is 'b'.
Thus, the imaginary part is $\mathrm{Im}(5+1i) = 1$.

Ex 20: Find the real part of the following complex number:

$$\mathrm{Re}((2+i)^2) = \boxed{3}$$

Answer: First, we must expand the expression to write it in the standard form $z = a + bi$.

$$\begin{aligned} (2+i)^2 &= (2)^2 + 2(2)(i) + (i)^2 \\ &= 4 + 4i + i^2 \\ &= 4 + 4i - 1 \\ &= 3 + 4i \end{aligned}$$

For a complex number in the form $z = a+bi$, the real part is 'a'.
Thus, the real part is $\mathrm{Re}(3+4i) = 3$.

Ex 21: Find the imaginary part of the following complex number:

$$\mathrm{Im}\left(\frac{3+i}{1-i}\right) = \boxed{2}$$

Answer: First, we must simplify the expression by multiplying the numerator and denominator by the conjugate of the denominator.

$$\begin{aligned} \frac{3+i}{1-i} &= \frac{3+i}{1-i} \times \frac{1+i}{1+i} \\ &= \frac{3(1) + 3(i) + i(1) + i(i)}{1^2 - i^2} \\ &= \frac{3 + 3i + i + i^2}{1-(-1)} \\ &= \frac{3 + 4i - 1}{2} \\ &= \frac{2+4i}{2} \\ &= 1 + 2i \end{aligned}$$

For a complex number in the form $z = a+bi$, the imaginary part is 'b'.
Thus, the imaginary part is $\mathrm{Im}(1+2i) = 2$.

B.4 CALCULATING POWERS OF THE IMAGINARY UNIT

Ex 22: Write in terms of i:

- $i^0 = \boxed{1}$
- $i^1 = \boxed{i}$
- $i^2 = \boxed{-1}$
- $i^3 = \boxed{-i}$

Answer:

- By convention, any non-zero number raised to the power of 0 is 1.
$$i^0 = 1$$
- Any number raised to the power of 1 is itself.
$$i^1 = i$$
- By definition of the imaginary unit.
$$i^2 = -1$$
- Using the properties of exponents:
$$i^3 = i^2 \times i = -1 \times i = -i$$

Ex 23: Prove that $i^{4n} = 1$ for n a natural number.

Answer:

- **Direct proof** Let n be a natural number.
$$\begin{aligned} i^{4n} &= \left(i^4\right)^n \\ &= \left(\left(i^2\right)^2\right)^n \\ &= \left((-1)^2\right)^n \\ &= (1)^n \\ &= 1 \end{aligned}$$
- **Induction proof** Let $P(n)$ be the statement "$i^{4n} = 1$" for $n \in \mathbb{N}$.

– **Base case:** For $n = 0, i^{4\times 0} = i^0$
$= 1$
Thus, $P(0)$ is true.

– **Inductive step:** Assume $P(k)$ is true for some integer $k \geq 0$. That is, $i^{4k} = 1$.We must show that $P(k+1)$ is true.

$$\begin{aligned} i^{4(k+1)} &= i^{4k+4} \\ &= i^{4k} \cdot i^4 \\ &= 1 \cdot (i^2)^2 \\ &= (-1)^2 \\ &= 1 \end{aligned}$$

Thus, $P(k+1)$ is true.

– **Conclusion:** Since $P(0)$ is true and $P(k+1)$ is true whenever $P(k)$ is true, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ex 24: Prove that $i^{4n+1} = i$ for n a natural number.

Answer:

- **Direct proof** Let n be a natural number.

$$\begin{aligned} i^{4n+1} &= i^{4n} \cdot i^1 \\ &= \left(i^4\right)^n \cdot i \\ &= (1)^n \cdot i \\ &= 1 \cdot i \\ &= i \end{aligned}$$

- **Induction proof** Let $P(n)$ be the statement "$i^{4n+1} = i$" for $n \in \mathbb{N}$.

  – **Base case:** For $n = 0, i^{4\times 0+1} = i^1$
  $= i$
  Thus, $P(0)$ is true.

  – **Inductive step:** Assume $P(k)$ is true for some integer $k \geq 0$. That is, $i^{4k+1} = i$.We must show that $P(k+1)$ is true.

$$\begin{aligned} i^{4(k+1)+1} &= i^{4k+4+1} \\ &= i^{4k+1} \cdot i^4 \\ &= i \cdot 1 \\ &= i \end{aligned}$$

  Thus, $P(k+1)$ is true.

  – **Conclusion:** Since $P(0)$ is true and $P(k+1)$ is true whenever $P(k)$ is true, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Ex 25: Write in terms of i:

- $i^{10} = \boxed{-1}$
- $i^{21} = \boxed{i}$
- $i^{400} = \boxed{1}$

Answer: Each time, perform the Euclidean division of the exponent by 4.

- $$\begin{aligned} i^{10} &= i^{4\times 2+2} \\ &= i^{4\times 2} \cdot i^2 \\ &= \left(i^4\right)^2 \cdot i^2 \\ &= (1)^2 \cdot (-1) \\ &= -1 \end{aligned}$$
- $$\begin{aligned} i^{21} &= i^{4\times 5+1} \\ &= i^{4\times 5} \cdot i^1 \\ &= \left(i^4\right)^5 \cdot i \\ &= (1)^5 \cdot i \\ &= i \end{aligned}$$
- $$\begin{aligned} i^{400} &= i^{4\times 100} \\ &= \left(i^4\right)^{100} \\ &= (1)^{100} \\ &= 1 \end{aligned}$$

B.5 EVALUATING POLYNOMIAL EXPRESSIONS OF A COMPLEX NUMBER

Ex 26: For $z = 1 + 2i$, write in standard form:

$$z^2 = \boxed{-3 + 4i}$$

Answer:

$$\begin{aligned} z^2 &= (1+2i)^2 \\ &= 1^2 + 2(1)(2i) + (2i)^2 \\ &= 1 + 4i + 4i^2 \\ &= 1 + 4i + 4(-1) \\ &= 1 + 4i - 4 \\ &= -3 + 4i \end{aligned}$$

Ex 27: For $z = 1 + i$, write in standard form:

$$z - z^2 = \boxed{1 - i}$$

Answer:

$$\begin{aligned} z - z^2 &= (1+i) - (1+i)^2 \\ &= (1+i) - (1^2 + 2i + i^2) \\ &= 1 + i - (1 + 2i - 1) \\ &= 1 + i - (2i) \\ &= 1 - i \end{aligned}$$

Ex 28: For $z = 1 + i$, write in standard form:

$$z^2 - z + 1 = \boxed{i}$$

Answer:

$$\begin{aligned} z^2 - z + 1 &= (1+i)^2 - (1+i) + 1 \\ &= (1 + 2i + i^2) - 1 - i + 1 \\ &= (1 + 2i - 1) - i \\ &= 2i - i \\ &= i \end{aligned}$$

Ex 29: For $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, write in standard form:

$$z^2 - z + 1 = \boxed{0}$$

Answer:

$$\begin{aligned}
z^2 - z + 1 &= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 1 \\
&= \left(\left(\frac{1}{2}\right)^2 + 2\frac{1}{2}\frac{\sqrt{3}}{2}i + \left(\frac{\sqrt{3}}{2}i\right)^2\right) - \frac{1}{2} - \frac{\sqrt{3}}{2}i + 1 \\
&= \left(\frac{1}{4} + \frac{\sqrt{3}}{2}i - \frac{3}{4}\right) - \frac{1}{2} - \frac{\sqrt{3}}{2}i + 1 \\
&= \left(-\frac{2}{4} + \frac{\sqrt{3}}{2}i\right) - \frac{1}{2} - \frac{\sqrt{3}}{2}i + 1 \\
&= -\frac{1}{2} + \frac{\sqrt{3}}{2}i - \frac{1}{2} - \frac{\sqrt{3}}{2}i + 1 \\
&= -1 + 1 \\
&= 0
\end{aligned}$$

C EQUALITY OF COMPLEX NUMBERS

C.1 SOLVING LINEAR EQUATIONS

Ex 30: Solve the equation $\frac{z+1}{z-1} = 2$ in $\mathbb{C}$.

Answer: First, we must ensure the denominator is not zero, so $z - 1 \neq 0$, which means $z \neq 1$.
To solve for z, we multiply both sides by $(z-1)$.

$$\begin{aligned}
\frac{z+1}{z-1} &= 2 \\
z + 1 &= 2(z-1) \\
z + 1 &= 2z - 2 \\
1 + 2 &= 2z - z \\
3 &= z
\end{aligned}$$

The solution is $z = 3$.

Ex 31: Solve the equation $z(1+i) = i$ in $\mathbb{C}$.

Answer: To solve for z, we divide by $(1+i)$.

$$\begin{aligned}
z(1+i) &= i \\
z &= \frac{i}{1+i} \\
&= \frac{i}{1+i} \times \frac{1-i}{1-i} \\
&= \frac{i(1-i)}{(1+i)(1-i)} \\
&= \frac{i - i^2}{1^2 - i^2} \\
&= \frac{i - (-1)}{1 - (-1)} \\
&= \frac{1+i}{2} \\
&= \frac{1}{2} + \frac{1}{2}i
\end{aligned}$$

The solution is $z = \frac{1}{2} + \frac{1}{2}i$.

Ex 32: Solve the equation $\frac{z+1}{z-1} = i$ in $\mathbb{C}$.

Answer: First, we must ensure the denominator is not zero, so $z - 1 \neq 0$, which means $z \neq 1$.
To solve for z, we multiply both sides by $(z-1)$.

$$\begin{aligned}
\frac{z+1}{z-1} &= i \\
z + 1 &= i(z-1) \\
z + 1 &= iz - i \\
z - iz &= -1 - i \\
z(1-i) &= -1 - i \\
z &= \frac{-1-i}{1-i}
\end{aligned}$$

To write this in standard form $a+bi$, we multiply the numerator and the denominator by the conjugate of the denominator, which is $1+i$.

$$\begin{aligned}
z &= \frac{-1-i}{1-i} \times \frac{1+i}{1+i} \\
&= \frac{-(1+i)(1+i)}{(1-i)(1+i)} \\
&= \frac{-(1+2i+i^2)}{1^2 - i^2} \\
&= \frac{-(1+2i-1)}{1-(-1)} \\
&= \frac{-2i}{2} \\
&= -i
\end{aligned}$$

The solution is $z = -i$.

C.2 SOLVING EQUATIONS BY EQUATING REAL AND IMAGINARY PARTS

Ex 33: For x, y real numbers, solve the equation $x(1+i) = 2y+1$.

Answer: First, we expand the left-hand side of the equation:

$$x(1+i) = x + xi$$

The right-hand side is a real number, so its imaginary part is zero. We can write it as $(2y+1) + 0i$.
Now we set the two expressions equal:

$$x + xi = (2y+1) + 0i$$

By equating the real and imaginary parts, we get a system of two equations:

$$\begin{cases} x = 2y + 1 \\ x = 0 \end{cases}$$

Substitute $x = 0$ into the first equation:

$$\begin{aligned}
0 &= 2y + 1 \\
-1 &= 2y \\
y &= -\frac{1}{2}
\end{aligned}$$

The solution is $x = 0$ and $y = -\frac{1}{2}$.

Ex 34: For x, y real numbers, solve the equation $(x+i)(2+i) = 1 + yi$.

Answer: First, we expand the left-hand side of the equation:

$$\begin{aligned}
(x+i)(2+i) &= 2x + xi + 2i + i^2 \\
&= 2x + xi + 2i - 1 \\
&= (2x-1) + (x+2)i
\end{aligned}$$

Now we set this equal to the right-hand side:

$$(2x-1)+(x+2)i = 1+yi$$

By equating the real and imaginary parts, we get a system of two equations:

$$\begin{cases} 2x-1=1 \\ x+2=y \end{cases}$$

Solving the first equation:

$$\begin{aligned} 2x-1 &= 1 \\ 2x &= 2 \\ x &= 1 \end{aligned}$$

Substitute $x=1$ into the second equation:

$$\begin{aligned} 1+2 &= y \\ y &= 3 \end{aligned}$$

The solution is $x=1$ and $y=3$.

Ex 35: For x, y real numbers, solve the equation $(x+2i)(1-i) = 2+yi$.

Answer: First, we expand the left-hand side of the equation:

$$\begin{aligned} (x+2i)(1-i) &= x - xi + 2i - 2i^2 \\ &= x - xi + 2i - 2(-1) \\ &= x - xi + 2i + 2 \\ &= (x+2) + (2-x)i \end{aligned}$$

Now we set this equal to the right-hand side:

$$(x+2)+(2-x)i = 2+yi$$

By equating the real and imaginary parts, we get a system of two equations:

$$\begin{cases} x+2=2 \\ 2-x=y \end{cases}$$

Solving the first equation:

$$\begin{aligned} x+2 &= 2 \\ x &= 0 \end{aligned}$$

Substitute $x=0$ into the second equation:

$$\begin{aligned} 2-0 &= y \\ y &= 2 \end{aligned}$$

The solution is $x=0$ and $y=2$.

D COMPLEX CONJUGATE

D.1 FINDING THE CONJUGATE OF A COMPLEX NUMBER

Ex 36: Find the conjugate of the following complex number:

$$\overline{1+i} = \boxed{1-i}$$

Answer: The conjugate of a complex number in standard form $a+bi$ is found by changing the sign of the imaginary part.
For $z=1+i$, the real part is 1 and the imaginary part is 1.
So, the conjugate is $\overline{z} = 1-i$.

Ex 37: Find the conjugate of the following complex number:

$$\overline{-i+1} = \boxed{1+i}$$

Answer: First, write the complex number in standard form $a+bi$.

$$-i+1 = 1-i$$

The conjugate of a complex number in standard form $a+bi$ is found by changing the sign of the imaginary part.
For $z=1-i$, the real part is 1 and the imaginary part is -1.
So, the conjugate is $\overline{z} = 1-(-1)i = 1+i$.

Ex 38: Find the conjugate of the following complex number:

$$\overline{\tfrac{2-3i}{2}} = \boxed{1+\frac{3}{2}i}$$

Answer: First, write the complex number in standard form $a+bi$.

$$\begin{aligned} \frac{2-3i}{2} &= \frac{2}{2} - \frac{3}{2}i \\ &= 1 - \frac{3}{2}i \end{aligned}$$

The conjugate of a complex number in standard form $a+bi$ is found by changing the sign of the imaginary part.
For $z = 1-\frac{3}{2}i$, the real part is 1 and the imaginary part is $-\frac{3}{2}$.
So, the conjugate is $\overline{z} = 1-(-\frac{3}{2})i = 1+\frac{3}{2}i$.

Ex 39: Find the conjugate of the following complex number:

$$\overline{2(1+i)} = \boxed{2-2i}$$

Answer: First, write the complex number in standard form $a+bi$.

$$2(1+i) = 2+2i$$

The conjugate of a complex number in standard form $a+bi$ is found by changing the sign of the imaginary part.
For $z=2+2i$, the real part is 2 and the imaginary part is 2.
So, the conjugate is $\overline{z} = 2-2i$.

D.2 PROVING PROPERTIES OF THE COMPLEX CONJUGATE

Ex 40: Given a complex number z, prove that $\overline{\overline{z}} = z$.

Answer: Let $z = a+bi$ with a and b real numbers.

$$\begin{aligned} \overline{\overline{z}} &= \overline{\overline{(a+bi)}} \\ &= \overline{a-bi} \\ &= a+bi \\ &= z \end{aligned}$$

Ex 41: Given two complex numbers z and w, prove that $\overline{z+w} = \overline{z}+\overline{w}$.

Answer: Let $z=a+bi$ and $w=c+di$ with a, b, c and d real numbers.

$$\begin{aligned} \overline{z+w} &= \overline{(a+bi)+(c+di)} \\ &= \overline{(a+c)+(b+d)i} \\ &= (a+c)-(b+d)i \\ &= a-bi+c-di \\ &= \overline{z}+\overline{w} \end{aligned}$$

Ex 42: Given two complex numbers z and w, prove that $\overline{z-w} = \overline{z}-\overline{w}$.

Answer: Let $z = a + bi$ and $w = c + di$ with a, b, c and d real numbers.

$$\begin{aligned}\overline{z-w} &= \overline{(a+bi)-(c+di)}\\ &= \overline{(a-c)+(b-d)i}\\ &= (a-c)-(b-d)i\\ &= a-c-bi+di\\ &= (a-bi)-(c-di)\\ &= \overline{z}-\overline{w}\end{aligned}$$

Ex 43: Given two complex numbers z and w, prove that $\overline{zw} = \overline{z} \cdot \overline{w}$.

Answer: Let $z = a + bi$ and $w = c + di$ with a, b, c and d real numbers.

$$\begin{aligned}\overline{zw} &= \overline{(a+bi)(c+di)}\\ &= \overline{ac+adi+bci+bdi^2}\\ &= \overline{(ac-bd)+(ad+bc)i}\\ &= (ac-bd)-(ad+bc)i\end{aligned}$$

And

$$\begin{aligned}\overline{z}\cdot\overline{w} &= (a-bi)(c-di)\\ &= ac-adi-bci+bdi^2\\ &= (ac-bd)-(ad+bc)i\end{aligned}$$

Thus, $\overline{zw} = \overline{z} \cdot \overline{w}$.

Ex 44: Given a complex number z, prove that $z + \overline{z}$ is a real number.

Answer: Let $z = a + bi$ with a and b real numbers.

$$\begin{aligned}z+\overline{z} &= (a+bi)+\overline{a+bi}\\ &= (a+bi)+(a-bi)\\ &= 2a\end{aligned}$$

Since a is a real number, $2a$ is a real number.
Thus $z + \overline{z}$ is a real number.

Ex 45: Given a complex number z, prove that $z \cdot \overline{z}$ is a non-negative real number.

Answer: Let $z = a + bi$ with a and b real numbers.

$$\begin{aligned}z\cdot\overline{z} &= (a+bi)\cdot(a-bi)\\ &= a^2-(bi)^2\\ &= a^2-b^2i^2\\ &= a^2-b^2(-1)\\ &= a^2+b^2\end{aligned}$$

Since a and b are real numbers, their squares, a^2 and b^2, are also real. The sum of two real numbers is real, so $a^2 + b^2$ is a real number.
Furthermore, since the square of a real number is always greater than or equal to zero, we have $a^2 \geq 0$ and $b^2 \geq 0$. The sum of two non-negative numbers is also non-negative, so $a^2 + b^2 \geq 0$.
Thus, $z \cdot \overline{z}$ is a non-negative real number.

D.3 SOLVING COMPLEX EQUATIONS INVOLVING THE CONJUGATE

Ex 46: Solve the equation $z = 2\overline{z} + 1$ in $\mathbb{C}$. (Hint: Let $z = a + bi$ where a and b are real numbers.)

Answer: Let $z = a + bi$, where a and b are real numbers. The equation becomes:

$$\begin{aligned}a+bi &= 2\overline{a+bi}+1\\ a+bi &= 2(a-bi)+1\\ a+bi &= 2a-2bi+1\\ a+bi &= (2a+1)-2bi\end{aligned}$$

By equating the real and imaginary parts, we get a system of two equations:

$$\begin{cases}a = 2a+1\\ b = -2b\end{cases}$$

Solving the first equation:

$$\begin{aligned}a &= 2a+1\\ -a &= 1\\ a &= -1\end{aligned}$$

Solving the second equation:

$$\begin{aligned}b &= -2b\\ 3b &= 0\\ b &= 0\end{aligned}$$

The solution is $z = a + bi = -1 + 0i = -1$.

Ex 47: Solve the equation $2z + 3\overline{z} = 1 + i$ in $\mathbb{C}$. (Hint: Let $z = a + bi$ where a and b are real numbers.)

Answer: Let $z = a + bi$, where a and b are real numbers. The equation becomes:

$$\begin{aligned}2(a+bi)+3\overline{a+bi} &= 1+i\\ 2a+2bi+3(a-bi) &= 1+i\\ 2a+2bi+3a-3bi &= 1+i\\ (2a+3a)+(2b-3b)i &= 1+i\\ 5a-bi &= 1+i\end{aligned}$$

By equating the real and imaginary parts, we get a system of two equations:

$$\begin{cases}5a = 1\\ -b = 1\end{cases}$$

Solving the system gives:

$$\begin{cases}a = \frac{1}{5}\\ b = -1\end{cases}$$

The solution is $z = a + bi = \frac{1}{5} - i$.

Ex 48: Solve the equation $(1 + i)z + \overline{z} = 2 - i$ in $\mathbb{C}$. (Hint: Let $z = a + bi$ where a and b are real numbers.)

Answer: Let $z = a + bi$, where a and b are real numbers. The equation becomes:

$$\begin{aligned}(1+i)(a+bi)+\overline{a+bi} &= 2-i\\ (a+bi+ai+bi^2)+(a-bi) &= 2-i\\ a+bi+ai-b+a-bi &= 2-i\\ (2a-b)+ai &= 2-i\end{aligned}$$

By equating the real and imaginary parts, we get a system of two equations:

$$\begin{cases}2a-b = 2\\ a = -1\end{cases}$$

Substitute $a = -1$ into the first equation:

$$\begin{aligned} 2(-1) - b &= 2 \\ -2 - b &= 2 \\ -b &= 4 \\ b &= -4 \end{aligned}$$

The solution is $z = a + bi = -1 - 4i$.