BINOMIAL EXPANSION

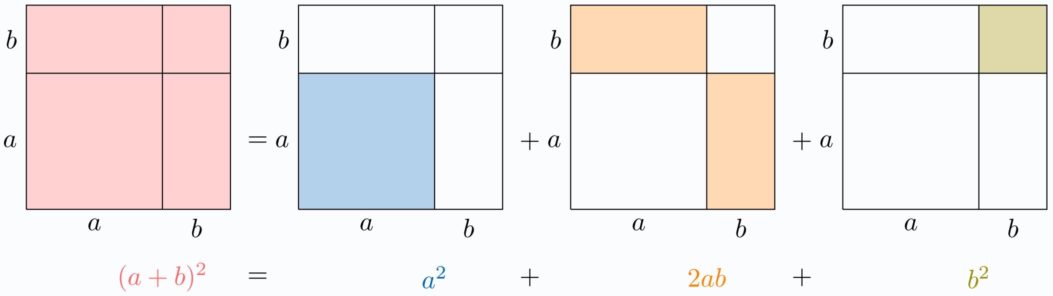
In this chapter we study the expansion of powers of a binomial expression such as $(a+b)^n$, where n is a positive integer. We will discover patterns in the coefficients using Pascal's triangle, and then state and use the **Binomial Theorem**.

A BINOMIAL EXPANSION FOR $n = 2$ AND $n = 3$

Proposition Perfect Squares Expansion

The square of a sum and the square of a difference can be written as:

$$(a+b)^2 = a^2 + 2ab + b^2 \quad \text{and} \quad (a-b)^2 = a^2 - 2ab + b^2.$$



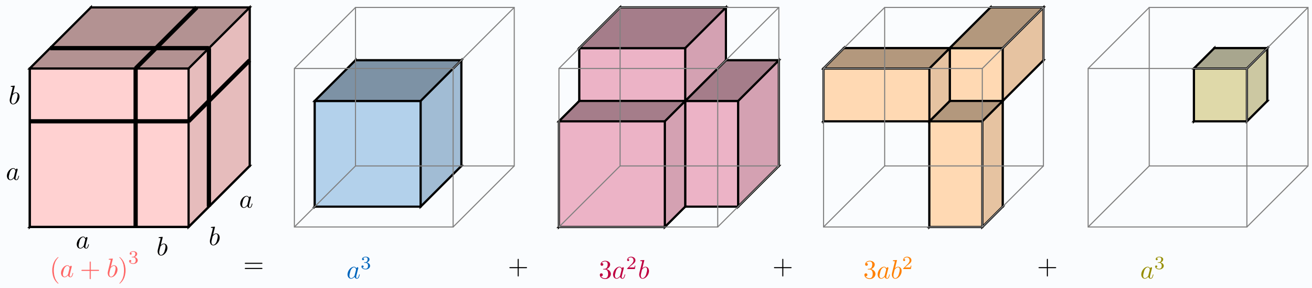
Ex: Expand and simplify $(x+2)^2$.

Answer: Using the formula $(a+b)^2 = a^2 + 2ab + b^2$ with $a = x$ and $b = 2$:

$$\begin{aligned}(x+2)^2 &= x^2 + 2\times x \times 2 + 2^2\\ &= x^2 + 4x + 4.\end{aligned}$$

So $(x+2)^2 = x^2 + 4x + 4$.

Proposition Perfect Cube Expansion



Ex: Expand and simplify $(x+2)^3$

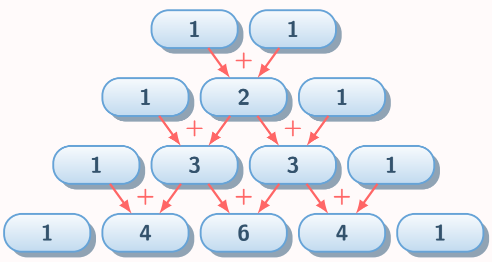
Answer: In the perfect cube expansion, we substitute $a = x$ and $b = 2$:

$$\begin{aligned}(x+2)^3 &= x^3 + 3\times x^2 \times 2 + 3 \times x \times 2^2 + 2^3\\ &= x^3 + 6x^2 + 12x + 8\end{aligned}$$

B PASCAL'S TRIANGLE

Definition Pascal's Triangle

- The values at the ends of each row are always 1.
- Each interior value is found by adding the two values diagonally above it.



Ex: Find the 5th row of Pascal's triangle.

Answer:

			1		1						row 1
		1		2		1					row 2
	1		3		3		1				row 3
1		4		6		4		1			row 4
1	5		10		10		5		1		row 5

So the 5th row is $1, 5, 10, 10, 5, 1$.

Proposition Binomial Expansion

For the binomial expansion of $(a+b)^n$ where $n \in \mathbb{N}$:

- As we look from left to right across the expansion, the powers of a decrease by 1, while the powers of b increase by 1.
- The sum of the powers of a and b in each term of the expansion is n.
- The number of terms in the expansion is $n+1$.
- The coefficients of the terms are row n of Pascal's triangle.

Ex: Find the binomial expansion of $(a+b)^5$.

Answer: From the 5th row of Pascal's triangle

			1		1						row 1
		1		2		1					row 2
	1		3		3		1				row 3
1		4		6		4		1			row 4
1	5		10		10		5		1		row 5

we get

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

C THE BINOMIAL THEOREM

Definition Factorial

For any positive integer n, $n!$ (read as "n factorial") is the product of the first n positive integers:

$$n! = n \times (n-1) \times \cdots \times 2 \times 1.$$

By convention, we define $0! = 1$.

Ex: Calculate $4!$.

Answer: $4! = 4 \times 3 \times 2 \times 1$
$= 24$

Definition Binomial Coefficient

For any integers $n \geqslant p \geqslant 0$, the **binomial coefficient** $\binom{n}{p}$ is defined as

$$\binom{n}{p} = \frac{n!}{p!(n-p)!}$$

Proposition Binomial Theorem

For any integer $n > 0$ and any real numbers $a, b \in \mathbb{R}$, we have

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^0 b^n,$$

or more compactly,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$