BINOMIAL EXPANSION

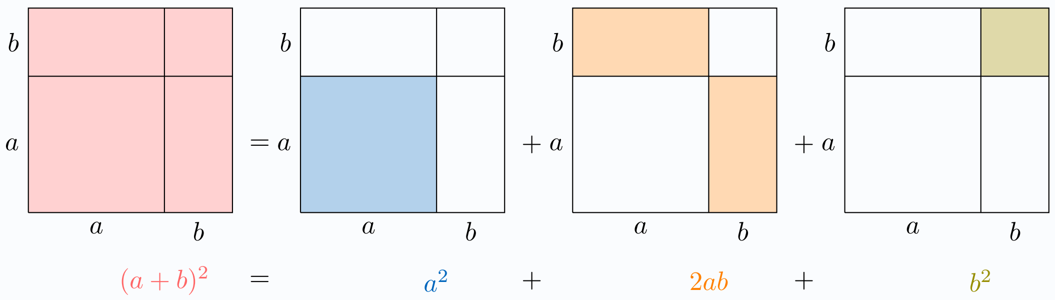
In this chapter we study the expansion of powers of a binomial expression such as $(a+b)^n$, where n is a positive integer. We will discover patterns in the coefficients using Pascal's triangle, and then state and use the **Binomial Theorem**.

A BINOMIAL EXPANSION FOR $n = 2$ AND $n = 3$

Proposition Perfect Squares Expansion

The square of a sum and the square of a difference can be written as:

$$(a+b)^2 = a^2 + 2ab + b^2 \quad \text{and} \quad (a-b)^2 = a^2 - 2ab + b^2.$$



Proof

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) && \text{(definition of a square)}\\ &= a(a+b) + b(a+b) && \text{(distributive law)}\\ &= a^2 + ab + ab + b^2 && \text{(expanding)}\\ &= a^2 + 2ab + b^2 && \text{(combining like terms).}\end{aligned}$$

Similarly,

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b) && \text{(definition of a square)}\\ &= a(a-b) - b(a-b) && \text{(distributive law)}\\ &= a^2 - ab - ab + b^2 && \text{(expanding)}\\ &= a^2 - 2ab + b^2 && \text{(combining like terms).}\end{aligned}$$

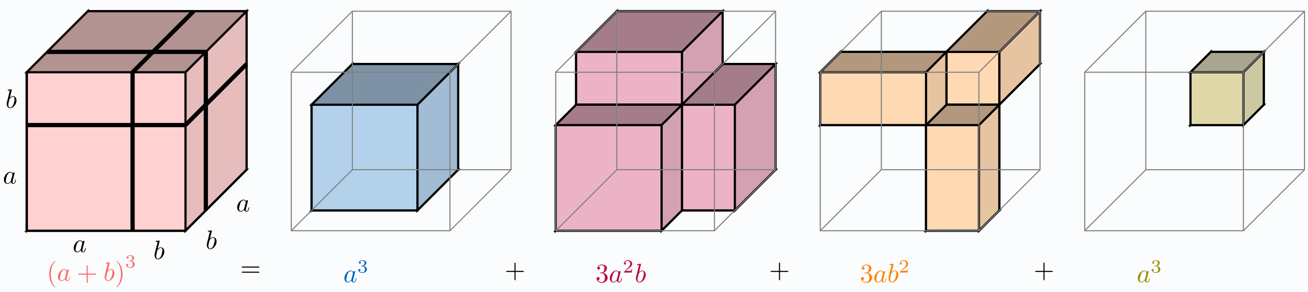
Ex: Expand and simplify $(x+2)^2$.

Answer: Using the formula $(a+b)^2 = a^2 + 2ab + b^2$ with $a = x$ and $b = 2$:

$$\begin{aligned}(x+2)^2 &= x^2 + 2 \times x \times 2 + 2^2\\ &= x^2 + 4x + 4.\end{aligned}$$

So $(x+2)^2 = x^2 + 4x + 4$.

Proposition Perfect Cube Expansion



Proof

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)(a+b) && \text{(cube definition)}\\ &= (a^2+2ab+b^2)(a+b) && \text{(using the square expansion)}\\ &= (a^2+2ab+b^2)a+(a^2+2ab+b^2)b && \text{(expanding)}\\ &= a^3+2a^2b+ab^2+a^2b+2ab^2+b^3 && \text{(distributive law)}\\ &= a^3+3a^2b+3ab^2+b^3 && \text{(combining)}\end{aligned}$$

Ex: Expand and simplify $(x+2)^3$

Answer: In the perfect cube expansion, we substitute $a=x$ and $b=2$:

$$\begin{aligned}(x+2)^3 &= x^3+3\times x^2\times 2+3\times x\times 2^2+2^3\\ &= x^3+6x^2+12x+8\end{aligned}$$

B PASCAL'S TRIANGLE

Discover: Consider the powers of $(a+b)$:

$$\begin{array}{lcccccccccc}(a+b)^1 &=& & & a & + & b & & & & \\ (a+b)^2 &=& & 1a^2 & + & 2ab & + & 1b^2 & & & \\ (a+b)^3 &=& 1a^3 & + & 3a^2b & + & 3ab^2 & + & 1b^3 & & \\ (a+b)^4 &=& 1a^4 & + & 4a^3b & + & 6a^2b^2 & + & 4ab^3 & + & 1b^4\end{array}$$

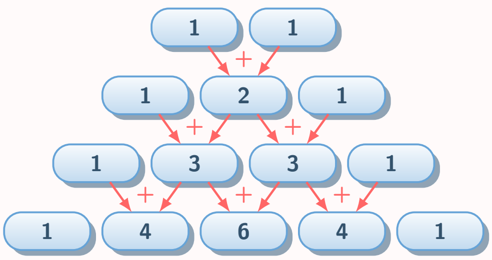
Now, when we list only the coefficients of the terms, we get Pascal's triangle:

			1		1				row 1
		1		2		1			row 2
	1		3		3		1		row 3
1		4		6		4		1	row 4

We observe the following pattern.

Definition **Pascal's Triangle**

- The values at the ends of each row are always 1.
- Each interior value is found by adding the two values diagonally above it.



Ex: Find the 5th row of Pascal's triangle.

Answer:

				1		1					row 1
			1		2		1				row 2
		1		3		3		1			row 3
	1		4		6		4		1		row 4
1		5		10		10		5		1	row 5

So the 5th row is $1, 5, 10, 10, 5, 1$.

Proposition **Binomial Expansion**

For the binomial expansion of $(a+b)^n$ where $n \in \mathbb{N}$:

- As we look from left to right across the expansion, the powers of a decrease by 1, while the powers of b increase by 1.
- The sum of the powers of a and b in each term of the expansion is n.

- The number of terms in the expansion is $n+1$.
- The coefficients of the terms are row n of Pascal's triangle.

Ex: Find the binomial expansion of $(a+b)^5$.

Answer: From the 5th row of Pascal's triangle

$$\begin{array}{ccccccccccccl} & & & 1 & & 1 & & & & & & & \text{row } 1 \\ & & 1 & & 2 & & 1 & & & & & & \text{row } 2 \\ & 1 & & 3 & & 3 & & 1 & & & & & \text{row } 3 \\ 1 & & 4 & & 6 & & 4 & & 1 & & & & \text{row } 4 \\ 1 & 5 & & 10 & & 10 & & 5 & & 1 & & & \text{row } 5 \end{array}$$

we get

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

C THE BINOMIAL THEOREM

Definition Factorial

For any positive integer n, $n!$ (read as "n factorial") is the product of the first n positive integers:

$$n! = n \times (n-1) \times \cdots \times 2 \times 1.$$

By convention, we define $0! = 1$.

Ex: Calculate $4!$.

Answer: $4! = 4 \times 3 \times 2 \times 1$
$= 24$

Definition Binomial Coefficient

For any integers $n \geqslant p \geqslant 0$, the **binomial coefficient** $\binom{n}{p}$ is defined as

$$\binom{n}{p} = \frac{n!}{p!(n-p)!}$$

Proposition Binomial Theorem

For any integer $n > 0$ and any real numbers $a, b \in \mathbb{R}$, we have

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^0 b^n,$$

or more compactly,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Proof

We give a combinatorial argument.
Consider the product

$$(a+b)^n = \underbrace{(a+b)(a+b)\cdots(a+b)}_{n \text{ factors}}.$$

To obtain a term in the expansion, we choose either a or b from each factor and multiply the n choices together.
A term of the form $a^{n-k}b^k$ appears whenever we choose b from exactly k of the n brackets (and a from the remaining $n-k$ brackets).
The number of such choices is precisely $\binom{n}{k}$, because we must choose which k positions will contribute a factor b.
Therefore the coefficient of $a^{n-k}b^k$ is $\binom{n}{k}$, and summing over all k from 0 to n gives

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$