BINOMIAL EXPANSION

A BINOMIAL EXPANSION FOR $n = 2$ AND $n = 3$

A.1 EXPANDING PERFECT SQUARE BINOMIALS (ADDITION)

Ex 1: Expand and simplify

$$(x+2)^2 = \boxed{x^2+4x+4}$$

Answer: In the perfect squares expansion, we substitute $a = x$ and $b = 2$:

$$\begin{aligned}(x+2)^2 &= x^2 + 2 \times x \times 2 + 2^2 \\ &= x^2 + 4x + 4\end{aligned}$$

Ex 2: Expand and simplify

$$(3+x)^2 = \boxed{9+6x+x^2}$$

Answer: In the perfect squares expansion, we substitute $a = 3$ and $b = x$:

$$\begin{aligned}(3+x)^2 &= 3^2 + 2 \times 3 \times x + x^2 \\ &= x^2 + 6x + 9\end{aligned}$$

Ex 3: Expand and simplify

$$(2x+1)^2 = \boxed{4x^2+4x+1}$$

Answer: In the perfect squares expansion, we substitute $a = 2x$ and $b = 1$:

$$\begin{aligned}(2x+1)^2 &= (2x)^2 + 2 \times 2x \times 1 + (1)^2 \\ &= 2^2x^2 + 4x + 1 \\ &= 4x^2 + 4x + 1\end{aligned}$$

Ex 4: Expand and simplify

$$(2+3x)^2 = \boxed{4+12x+9x^2}$$

Answer: In the perfect squares expansion, we substitute $a = 2$ and $b = 3x$:

$$\begin{aligned}(2+3x)^2 &= 2^2 + 2 \times 2 \times 3x + (3x)^2 \\ &= 4 + 12x + 3^2x^2 \\ &= 9x^2 + 12x + 4\end{aligned}$$

A.2 EXPANDING PERFECT SQUARE BINOMIALS (SUBTRACTION)

Ex 5: Expand and simplify

$$(x-2)^2 = \boxed{x^2-4x+4}$$

Answer: In the perfect squares expansion, we substitute $a = x$ and $b = 2$:

$$\begin{aligned}(x-2)^2 &= x^2 - 2 \times x \times 2 + 2^2 \\ &= x^2 - 4x + 4\end{aligned}$$

Ex 6: Expand and simplify

$$(3-x)^2 = \boxed{9-6x+x^2}$$

Answer: In the perfect squares expansion, we substitute $a = 3$ and $b = x$:

$$\begin{aligned}(3-x)^2 &= 3^2 - 2 \times 3 \times x + x^2 \\ &= x^2 - 6x + 9\end{aligned}$$

Ex 7: Expand and simplify

$$(2x-1)^2 = \boxed{4x^2-4x+1}$$

Answer: In the perfect squares expansion, we substitute $a = 2x$ and $b = 1$:

$$\begin{aligned}(2x-1)^2 &= (2x)^2 - 2 \times 2x \times 1 + (1)^2 \\ &= 4x^2 - 4x + 1\end{aligned}$$

Ex 8: Expand and simplify

$$(2-3x)^2 = \boxed{4-12x+9x^2}$$

Answer: In the perfect squares expansion, we substitute $a = 2$ and $b = 3x$:

$$\begin{aligned}(2-3x)^2 &= 2^2 - 2 \times 2 \times 3x + (3x)^2 \\ &= 9x^2 - 12x + 4\end{aligned}$$

A.3 EXPANDING PERFECT CUBE BINOMIALS (ADDITION)

Ex 9: Expand and simplify

$$(x+1)^3 = \boxed{x^3+3x^2+3x+1}$$

Answer: In the perfect cubes expansion, we substitute $a = x$ and $b = 1$:

$$\begin{aligned}(x+1)^3 &= x^3 + 3 \times x^2 \times 1 + 3 \times x \times 1^2 + 1^3 \\ &= x^3 + 3x^2 + 3x + 1\end{aligned}$$

Ex 10: Expand and simplify

$$(x+3)^3 = \boxed{x^3+9x^2+27x+27}$$

Answer: In the perfect cubes expansion, we substitute $a = x$ and $b = 3$:

$$\begin{aligned}(x+3)^3 &= x^3 + 3 \times x^2 \times 3 + 3 \times x \times 3^2 + 3^3 \\ &= x^3 + 9x^2 + 27x + 27\end{aligned}$$

Ex 11: Expand and simplify

$$(2x+2)^3 = \boxed{8x^3+24x^2+24x+8}$$

Answer: In the perfect cubes expansion, we substitute $a = 2x$ and $b = 2$:

$$\begin{aligned}(2x+2)^3 &= (2x)^3 + 3 \times (2x)^2 \times 2 + 3 \times 2x \times (2)^2 + (2)^3 \\ &= 8x^3 + 24x^2 + 24x + 8\end{aligned}$$

Ex 12: Expand and simplify

$$(2x+3)^3 = \boxed{8x^3+36x^2+54x+27}$$

Answer: In the perfect cubes expansion, we substitute $a = 2x$ and $b = 3$:

$$\begin{aligned}(2x+3)^3 &= (2x)^3 + 3 \times (2x)^2 \times 3 + 3 \times 2x \times (3)^2 + (3)^3 \\ &= 8x^3 + 36x^2 + 54x + 27\end{aligned}$$

A.4 EXPANDING PERFECT CUBE BINOMIALS (SUBTRACTION)

Ex 13: Expand and simplify

$$(x-1)^3 = \boxed{x^3 - 3x^2 + 3x - 1}$$

Answer: In the perfect cubes expansion, we substitute $a = x$ and $b = (-1)$:

$$\begin{aligned}(x + (-1))^3 &= x^3 + 3 \times x^2 \times (-1) + 3 \times x \times ((-1))^2 + ((-1))^3 \\ &= x^3 - 3x^2 + 3x - 1\end{aligned}$$

Ex 14: Expand and simplify

$$(x-2)^3 = \boxed{x^3 - 6x^2 + 12x - 8}$$

Answer: In the perfect cubes expansion, we substitute $a = x$ and $b = (-2)$:

$$\begin{aligned}(x + (-2))^3 &= x^3 + 3 \times x^2 \times (-2) + 3 \times x \times (-2)^2 + (-2)^3 \\ &= x^3 - 6x^2 + 12x - 8\end{aligned}$$

Ex 15: Expand and simplify

$$(3-x)^3 = \boxed{27 - 27x + 9x^2 - x^3}$$

Answer: In the perfect cubes expansion, we substitute $a = 3$ and $b = (-x)$:

$$\begin{aligned}(3 + (-x))^3 &= 3^3 + 3 \times 3^2 \times (-x) + 3 \times 3 \times ((-x))^2 + ((-x))^3 \\ &= 27 - 27x + 9x^2 - x^3\end{aligned}$$

Ex 16: Expand and simplify

$$(2x-1)^3 = \boxed{8x^3 - 12x^2 + 6x - 1}$$

Answer: In the perfect cubes expansion, we substitute $a = 2x$ and $b = (-1)$:

$$\begin{aligned}(2x + (-1))^3 &= (2x)^3 + 3 \times (2x)^2 \times (-1) + 3 \times 2x \times ((-1))^2 + ((-1))^3 \\ &= 8x^3 - 12x^2 + 6x - 1\end{aligned}$$

B PASCAL'S TRIANGLE

B.1 BUILDING PASCAL'S TRIANGLE

Ex 17: Write down the first five rows of Pascal's triangle.

Answer:

		1	1			row 1
	1	2	1			row 2
	1	3	3	1		row 3
1	4	6	4	1		row 4
1	5	10	10	5	1	row 5

Ex 18: Write down the first six rows of Pascal's triangle.

Answer:

			1	1			row 1
		1	2	1			row 2
		1	3	3	1		row 3
	1	4	6	4	1		row 4
	1	5	10	10	5	1	row 5
1	6	15	20	15	6	1	row 6

B.2 USING THE FOURTH ROW OF PASCAL'S TRIANGLE

Ex 19: Use Pascal's triangle to find the coefficients of the expansion of $(a+b)^4$, then expand and simplify.

Answer: From the 4th row of Pascal's triangle, the coefficients are

$$1,\ 4,\ 6,\ 4,\ 1.$$

So

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Ex 20: Use Pascal's triangle to expand and simplify:

$$(x-1)^4 = \boxed{x^4 - 4x^3 + 6x^2 - 4x + 1}$$

Answer: From the 4th row of Pascal's triangle, the coefficients are

$$1,\ 4,\ 6,\ 4,\ 1.$$

So

$$(x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1.$$

Ex 21: Use Pascal's triangle to expand and simplify

$$(2x-1)^4 = \boxed{16x^4 - 32x^3 + 24x^2 - 8x + 1}$$

Answer: Using the coefficients $1, 4, 6, 4, 1$ from the 4th row:

$$\begin{aligned}(2x-1)^4 &= 1 \cdot (2x)^4 + 4 \cdot (2x)^3(-1) + 6 \cdot (2x)^2(-1)^2 \\ &\quad + 4 \cdot (2x)(-1)^3 + 1 \cdot (-1)^4 \\ &= 16x^4 - 32x^3 + 24x^2 - 8x + 1.\end{aligned}$$

B.3 USING THE FIFTH ROW OF PASCAL'S TRIANGLE

Ex 22: Use Pascal's triangle to find the coefficients of the expansion of $(a+b)^5$, then expand and simplify.

Answer:

		1	1			row 1
	1	2	1			row 2
	1	3	3	1		row 3
1	4	6	4	1		row 4
1	5	10	10	5	1	row 5

From the 5th row of Pascal's triangle, the coefficients are

$$1,\ 5,\ 10,\ 10,\ 5,\ 1.$$

So

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Ex 23: Use Pascal's triangle to expand and simplify:

$$(x+1)^5 = \boxed{x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1}$$

Answer: From the 5th row of Pascal's triangle, the coefficients are

$$1,\ 5,\ 10,\ 10,\ 5,\ 1.$$

So

$$(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1.$$

Ex 24: Use Pascal's triangle to expand and simplify:

$$(x-1)^5 = \boxed{x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1}$$

Answer: From the 5th row of Pascal's triangle, the coefficients are

$$1,\ 5,\ 10,\ 10,\ 5,\ 1.$$

Because of the minus sign in $(x-1)$, the signs of the terms will alternate:

$$\begin{aligned}(x-1)^5 &= 1(x)^5(-1)^0 + 5(x)^4(-1)^1 + 10(x)^3(-1)^2 + 10(x)^2(-1)^3 + 5(x)^1(-1)^4 + 1(x)^0(-1)^5 \\ &= x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1.\end{aligned}$$

Ex 25: Use Pascal's triangle to expand and simplify:

$$(2-x)^5 = \boxed{-x^5 + 10x^4 - 40x^3 + 80x^2 - 80x + 32}$$

Answer: From the 5th row of Pascal's triangle, the coefficients are

$$1,\ 5,\ 10,\ 10,\ 5,\ 1.$$

So

$$(2-x)^5 = 2^5 - 5\cdot 2^4x + 10\cdot 2^3x^2 - 10\cdot 2^2x^3 + 5\cdot 2x^4 - x^5,$$

which simplifies to

$$(2-x)^5 = -x^5 + 10x^4 - 40x^3 + 80x^2 - 80x + 32.$$

C THE BINOMIAL THEOREM

C.1 EVALUATING FACTORIALS WITHOUT A CALCULATOR

Ex 26: Evaluate:

$$3! = \boxed{6}.$$

Answer:

$$\begin{aligned}3! &= 3\times 2\times 1 \\ &= 6\end{aligned}$$

Ex 27: Evaluate:

$$4! = \boxed{24}$$

Answer:

$$\begin{aligned}4! &= 4\times 3\times 2\times 1 \\ &= 24\end{aligned}$$

Ex 28: Evaluate:

$$5! = \boxed{120}$$

Answer:

$$\begin{aligned}5! &= 5\times 4\times 3\times 2\times 1 \\ &= 120\end{aligned}$$

Ex 29: Evaluate:

$$6! = \boxed{720}$$

Answer:

$$\begin{aligned}6! &= 6\times 5\times 4\times 3\times 2\times 1 \\ &= 720\end{aligned}$$

C.2 EVALUATING FACTORIALS WITH A CALCULATOR

Ex 30: Evaluate:

$$7! = \boxed{5040}$$

Answer: Enter $7!$ in the calculator.

Ex 31: Evaluate:

$$\frac{8!}{3!} = \boxed{6720}$$

Answer: Enter $\frac{8!}{3!}$ in the calculator.

Ex 32: Evaluate:

$$\frac{9!}{3!6!} = \boxed{84}$$

Answer: Enter $\frac{9!}{3!\times 6!}$ in the calculator.

Ex 33: Evaluate:

$$\binom{20}{17} = \boxed{1140}$$

Answer:

- $\binom{20}{17} = \dfrac{20!}{17!(20-17)!} = \dfrac{20!}{17!3!}$
- Enter $\dfrac{20!}{17!3!}$ in the calculator.

Ex 34: Evaluate:

$$\binom{15}{10} = \boxed{3003}$$

Answer:

- $\binom{15}{10} = \dfrac{15!}{10!(15-10)!} = \dfrac{15!}{10!5!}$
- Enter $\dfrac{15!}{10!5!}$ in the calculator.

C.3 EXPRESSING PRODUCTS IN FACTORIAL FORM

Ex 35: Express in factorial form:

$$\frac{4\times 3\times 2\times 1}{2\times 1} = \boxed{\frac{4!}{2!}}$$

Answer:

$$\frac{4\times 3\times 2\times 1}{2\times 1} = \frac{4!}{2!}$$

Ex 36: Express in factorial form:

$$4\times 3 = \boxed{\frac{4!}{2!}}$$

Answer:

$$\begin{aligned} 4 \times 3 &= \frac{4 \times 3 \times (2 \times 1)}{(2 \times 1)} \\ &= \frac{4!}{2!} \end{aligned}$$

Ex 37: Express in factorial form:

$$\frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \boxed{\frac{5!}{3!}}$$

Answer:

$$\frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{5!}{3!}$$

Ex 38: Express in factorial form:

$$5 \times 4 = \boxed{\frac{5!}{3!}}$$

Answer:

$$\begin{aligned} 5 \times 4 &= \frac{5 \times 4 \times (3 \times 2 \times 1)}{(3 \times 2 \times 1)} \\ &= \frac{5!}{3!} \end{aligned}$$

Ex 39: Express in factorial form:

$$\frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = \boxed{\frac{7!}{4!}}$$

Answer:

$$\frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = \frac{7!}{4!}$$

Ex 40: Express in factorial form:

$$7 \times 6 \times 5 = \boxed{\frac{7!}{4!}}$$

Answer:

$$\begin{aligned} 7 \times 6 \times 5 &= \frac{7 \times 6 \times 5 \times (4 \times 3 \times 2 \times 1)}{(4 \times 3 \times 2 \times 1)} \\ &= \frac{7!}{4!} \end{aligned}$$

C.4 EVALUATING FACTORIALS BY SIMPLIFICATION

Ex 41: Evaluate

$$\binom{5}{3} = \boxed{10}$$

Answer:

$$\begin{aligned} \binom{5}{3} &= \frac{5!}{3!(5-3)!} \\ &= \frac{5!}{3!2!} \\ &= \frac{5 \times 4 \times \cancel{3 \times 2 \times 1}}{\cancel{3 \times 2 \times 1} \times 2 \times 1} \\ &= \frac{5 \times 4}{2 \times 1} \\ &= 10 \end{aligned}$$

Ex 42: Evaluate

$$\binom{6}{4} = \boxed{15}$$

Answer:

$$\begin{aligned} \binom{6}{4} &= \frac{6!}{4!(6-4)!} \\ &= \frac{6!}{4!2!} \\ &= \frac{6 \times 5 \times \cancel{4 \times 3 \times 2 \times 1}}{\cancel{4 \times 3 \times 2 \times 1} \times 2 \times 1} \\ &= \frac{6 \times 5}{2 \times 1} \\ &= 15 \end{aligned}$$

Ex 43: Evaluate

$$\binom{7}{2} = \boxed{21}$$

Answer:

$$\begin{aligned} \binom{7}{2} &= \frac{7!}{2!(7-2)!} \\ &= \frac{7!}{2!5!} \\ &= \frac{7 \times 6 \times \cancel{5 \times 4 \times 3 \times 2 \times 1}}{2 \times 1 \times \cancel{5 \times 4 \times 3 \times 2 \times 1}} \\ &= \frac{7 \times 6}{2 \times 1} \\ &= 21 \end{aligned}$$

Ex 44: Evaluate

$$\binom{7}{4} = \boxed{35}$$

Answer:

$$\begin{aligned} \binom{7}{4} &= \frac{7!}{4!(7-4)!} \\ &= \frac{7!}{4!3!} \\ &= \frac{7 \times 6 \times 5 \times \cancel{4 \times 3 \times 2 \times 1}}{\cancel{4 \times 3 \times 2 \times 1} \times 3 \times 2 \times 1} \\ &= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \\ &= \frac{7 \times \cancel{6} \times 5}{\cancel{6}} \\ &= 7 \times 5 \\ &= 35 \end{aligned}$$

C.5 FINDING A SPECIFIC TERM IN AN EXPANSION

Ex 45: Find the coefficient of x^3 in the expansion of $(x-1)^6$.

$$\boxed{-20}$$

Answer: From the binomial theorem,

$$(x-1)^6 = \sum_{k=0}^{6} \binom{6}{k} x^{6-k}(-1)^k.$$

The general term is

$$T_k = \binom{6}{k} x^{6-k}(-1)^k.$$

In this term, the power of x is $6-k$. We want $6-k=3$, so $k=3$.Thus the term in x^3 is

$$T_3 = \binom{6}{3} x^3(-1)^3 = 20\cdot x^3\cdot(-1) = -20x^3.$$

So the coefficient of x^3 is -20.

Ex 46: Find the coefficient of x^5 in the expansion of $(2x+3)^7$.

$\boxed{6048}$

Answer: From the binomial theorem,

$$(2x+3)^7 = \sum_{k=0}^{7}\binom{7}{k}(2x)^{7-k}3^k$$

The general term is

$$T_k = \binom{7}{k}(2x)^{7-k}3^k.$$

In this term, the power of x is $7-k$. We want $7-k=5$, so $k=2$.Thus the term in x^5 is

$$\begin{aligned} T_2 &= \binom{7}{2}(2x)^5 3^2 \\ &= \binom{7}{2}\cdot 2^5\cdot 3^2\, x^5 \\ &= 21\cdot 32\cdot 9\, x^5 \\ &= 6048x^5 \end{aligned}$$

So the coefficient of x^5 is 6048.

Ex 47: Find the constant term (term independent of x) in the expansion of $\left(x+\dfrac{1}{x}\right)^6$.

$\boxed{20}$

Answer: From the binomial theorem,

$$\left(x+\frac{1}{x}\right)^6 = \sum_{k=0}^{6}\binom{6}{k}x^{6-k}\left(\frac{1}{x}\right)^k.$$

The general term is

$$\begin{aligned} T_k &= \binom{6}{k}x^{6-k}\left(\frac{1}{x}\right)^k \\ &= \binom{6}{k}x^{6-k}\frac{1}{x^k} \\ &= \binom{6}{k}x^{6-k}x^{-k} \\ &= \binom{6}{k}x^{6-k-k} \\ &= \binom{6}{k}x^{6-2k} \end{aligned}$$

The exponent of x is $6-2k$. For the constant term we need $6-2k=0$, so $k=3$.
Thus the constant term is

$$\begin{aligned} T_3 &= \binom{6}{3}x^{6-2\cdot 3} \\ &= \binom{6}{3}x^0 \\ &= 20 \end{aligned}$$

So the constant term in the expansion is 20.

Ex 48: Find the coefficient of x^4 in the expansion of $(3x-2)^7$.

$\boxed{-22680}$

Answer: From the binomial theorem,

$$(3x-2)^7 = \sum_{k=0}^{7}\binom{7}{k}(3x)^{7-k}(-2)^k.$$

The general term is

$$T_k = \binom{7}{k}(3x)^{7-k}(-2)^k.$$

In this term, the power of x is $7-k$. We want $7-k=4$, so $k=3$.Thus the term in x^4 is

$$\begin{aligned} T_3 &= \binom{7}{3}(3x)^4(-2)^3 \\ &= \binom{7}{3}\cdot 3^4\cdot(-2)^3\, x^4 \\ &= 35\cdot 81\cdot(-8)\, x^4 \\ &= -22680x^4. \end{aligned}$$

So the coefficient of x^4 is -22680.

Ex 49: Find the constant term (term independent of x) in the expansion of $\left(2x-\dfrac{1}{x}\right)^6$.

$\boxed{-160}$

Answer: From the binomial theorem,

$$\left(2x-\frac{1}{x}\right)^6 = \sum_{k=0}^{6}\binom{6}{k}(2x)^{6-k}\left(-\frac{1}{x}\right)^k.$$

The general term is

$$\begin{aligned} T_k &= \binom{6}{k}(2x)^{6-k}\left(-\frac{1}{x}\right)^k \\ &= \binom{6}{k}2^{6-k}(-1)^k x^{6-k}x^{-k} \\ &= \binom{6}{k}2^{6-k}(-1)^k x^{6-2k}. \end{aligned}$$

The exponent of x is $6-2k$. For the constant term we need $6-2k=0$, so $k=3$.Thus the constant term is

$$\begin{aligned} T_3 &= \binom{6}{3}2^{6-3}(-1)^3x^{6-2\cdot 3} \\ &= \binom{6}{3}2^3(-1)^3x^0 \\ &= 20\cdot 8\cdot(-1) \\ &= -160. \end{aligned}$$

So the constant term in the expansion is -160.

C.6 DERIVING IDENTITIES FROM THE BINOMIAL THEOREM

Ex 50:

1. Show that
$$(1+x)^n = \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^n.$$

2. Hence deduce that:
   (a) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n-1}+\binom{n}{n}=2^n$
   (b) $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\ldots+(-1)^n\binom{n}{n}=0$

Answer:

1. From the binomial theorem (with $a=1$, $b=x$), we have
$$\begin{aligned}(1+x)^n &= \sum_{k=0}^{n}\binom{n}{k}1^{n-k}x^k\\ &= \sum_{k=0}^{n}\binom{n}{k}x^k\\ &= \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n\end{aligned}$$

2. (a) Substituting $x=1$ gives
$$\begin{aligned}(1+1)^n &= \binom{n}{0}+\binom{n}{1}1+\binom{n}{2}1^2+\ldots+\binom{n}{n}1^n\\ 2^n &= \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}\end{aligned}$$

   (b) Substituting $x=-1$ gives
$$\begin{aligned}(1-1)^n &= \binom{n}{0}+\binom{n}{1}(-1)+\binom{n}{2}(-1)^2+\ldots+\binom{n}{n}(-1)^n\\ (0)^n &= \binom{n}{0}-\binom{n}{1}+\binom{n}{2}+\ldots+(-1)^n\binom{n}{n}\\ 0 &= \binom{n}{0}-\binom{n}{1}+\binom{n}{2}+\ldots+(-1)^n\binom{n}{n}\end{aligned}$$

Ex 51:

1. By differentiating the binomial expansion of $(1+x)^n$, show that
$$n(1+x)^{n-1}=\sum_{k=1}^{n}k\binom{n}{k}x^{k-1}.$$

2. Hence, deduce that:
$$\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\ldots+n\binom{n}{n}=n2^{n-1}$$

Answer:

1. We start with the binomial expansion:
$$(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n$$
Differentiating both sides with respect to x:
$$\begin{aligned}\frac{d}{dx}(1+x)^n &= \frac{d}{dx}\left(\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n\right)\\ n(1+x)^{n-1} &= 0+\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+\ldots+n\binom{n}{n}x^{n-1}\\ n(1+x)^{n-1} &= \sum_{k=1}^{n}k\binom{n}{k}x^{k-1}\end{aligned}$$

2. To find the required sum, we substitute $x=1$ into the identity from part (1):
$$\begin{aligned}n(1+1)^{n-1} &= \sum_{k=1}^{n}k\binom{n}{k}(1)^{k-1}\\ n2^{n-1} &= \binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\ldots+n\binom{n}{n}\end{aligned}$$