APPLICATIONS OF INTEGRATION IN GEOMETRY

A CALCULATING GEOMETRIC AREA

In the previous chapter, we defined the definite integral $\int_a^b f(x)\,dx$ as the *signed area* between the graph of $y = f(x)$ and the x-axis. This means that areas above the x-axis are positive, while areas below are negative.
However, when a problem asks for the *geometric area* or simply *the area* of a region, it refers to the physical, positive space the region occupies. In this case, we must ensure that all parts of the region, whether they are above or below the x-axis, contribute a positive value to the total. This section outlines a method for calculating this total geometric area.

Method Calculating Geometric Area Bounded by a Curve and the x-axis

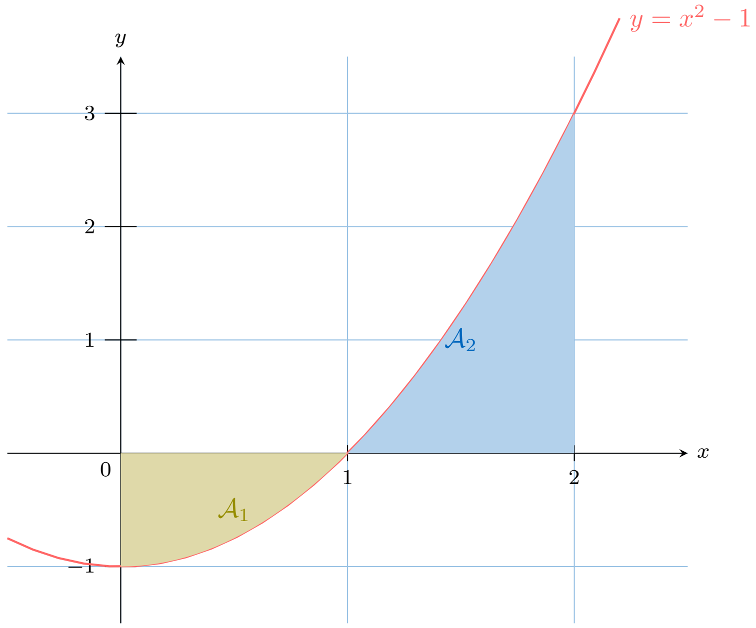
To find the total geometric area $\mathcal{A}$ bounded by a curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$:

1. **Find the x-intercepts:** Determine where the function crosses the x-axis by solving $f(x) = 0$. Identify any roots $c_1, c_2, \ldots$ that lie within the interval $[a, b]$.
2. **Split the integral:** Divide the main integral into smaller integrals at each intercept found in the previous step.
3. **Calculate each definite integral:** Compute the integral for each sub-interval. Some of these will correspond to positive values (where $f(x) \geq 0$) and some to negative values (where $f(x) \leq 0$).
4. **Sum the absolute values:** The total geometric area is the sum of the absolute values of these integrals. If the integral of a sub-region is negative, take its positive value before adding it to the total.

Ex: Find the total geometric area between the curve $y = x^2 - 1$ and the x-axis from $x = 0$ to $x = 2$.

Answer:

1. **Find intercepts:** We solve $f(x) = x^2 - 1 = 0$, which gives roots at $x = 1$ and $x = -1$. The only root within our interval of integration $[0, 2]$ is $x = 1$.
2. **Split the integral:** We must split the total area calculation at $x = 1$.
   - From $x = 0$ to $x = 1$, the function is below the x-axis (Area $\mathcal{A}_1$).
   - From $x = 1$ to $x = 2$, the function is above the x-axis (Area $\mathcal{A}_2$).



3. **Calculate each integral:**

$$\int_0^1 (x^2 - 1)\,dx = \left[\frac{x^3}{3} - x\right]_0^1 = \left(\frac{1}{3} - 1\right) - 0 = -\frac{2}{3}$$

$$\int_1^2 (x^2 - 1)\,dx = \left[\frac{x^3}{3} - x\right]_1^2 = \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right) = \frac{2}{3} - \left(-\frac{2}{3}\right) = \frac{4}{3}$$

4. **Sum the absolute values:**

$$\text{Total Area} = \left|-\frac{2}{3}\right| + \left|\frac{4}{3}\right| = \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = 2$$

The total geometric area is 2 square units.

Definition Geometric Area Formula

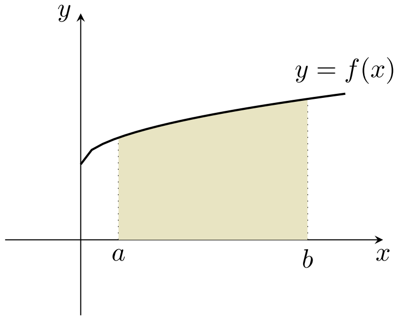
The **total geometric area** $\mathcal{A}$ between the graph of a function $f(x)$ and the x-axis from $x = a$ to $x = b$ is given by the integral of the absolute value of the function:

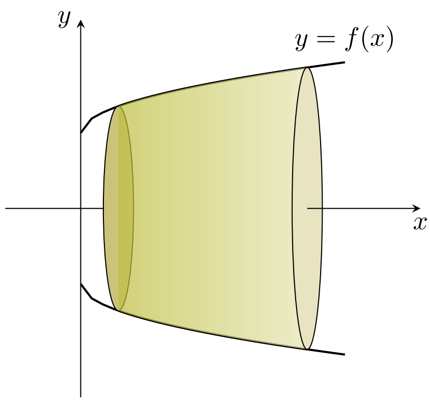
$$\mathcal{A} = \int_a^b |f(x)| \, dx$$

Note The method of splitting the integral and summing the absolute values is equivalent to this formal definition of the total geometric area.

B VOLUMES OF REVOLUTION

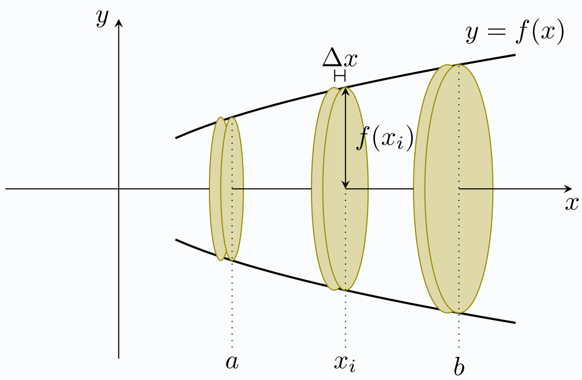
A **solid of revolution** is a three-dimensional object formed by rotating a two-dimensional shape around an axis. In this section, we will develop a method to find the exact volume of such solids.
Consider the area under the curve $y = f(x)$ from $x = a$ to $x = b$. If we revolve this area 360° (2π radians) around the x-axis, it sweeps out a solid of revolution.





Method The Disk Method

To find the volume of this solid, we use the same strategy as for area: slice the solid into many thin pieces and sum their volumes. In this case, each slice is a thin cylindrical disk. The key insight is that each disk is formed by rotating one of the thin rectangles from a Riemann sum around the x-axis.



The volume of a single cylinder is $\pi r^2 h$. For a disk at position x_i with a small thickness Δx:

- The **radius** is the height of the function, $r = f(x_i)$.
- The **height** (thickness) of the disk is $h = \Delta x$.

The volume of one disk is $V_i = \pi[f(x_i)]^2 \Delta x$. The total volume is approximated by summing the volumes of all the disks:

$$\begin{aligned}\mathcal{V} &\approx \sum_{i=0}^{n-1} V_i \\ &\approx \sum_{i=0}^{n-1} \pi[f(x_i)]^2 \Delta x\end{aligned}$$

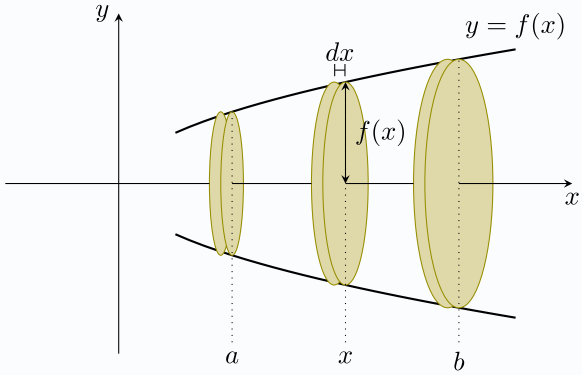
To find the volume *exactly*, we take the limit as the number of disks goes to infinity and their thickness approaches zero ($\Delta x \to 0$). This limit is the definite integral:

$$\begin{aligned}\mathcal{V} &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \pi[f(x_i)]^2 \Delta x \\ &= \int_a^b \pi[f(x)]^2\, dx \\ &= \pi \int_a^b [f(x)]^2\, dx\end{aligned}$$

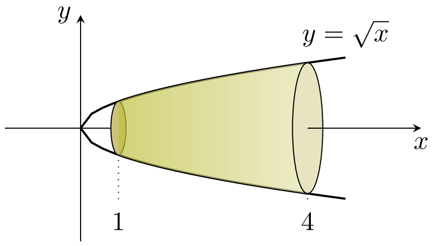
Proposition **Volume of Revolution about the x-axis**

Assuming $f(x) \geq 0$ on $[a, b]$, the volume V generated by rotating the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ around the x-axis is given by:

$$V = \pi \int_a^b [f(x)]^2\, dx$$



Ex: Find the volume of the solid generated by revolving the region under the curve $y = \sqrt{x}$ from $x = 1$ to $x = 4$ around the x-axis.



Answer:

1. **Identify:** The function is $f(x) = \sqrt{x}$, and the limits are $a = 1$ and $b = 4$.
2. **Square the function:** $[f(x)]^2 = (\sqrt{x})^2 = x$.
3. **Integrate:** We set up and evaluate the integral for the volume:

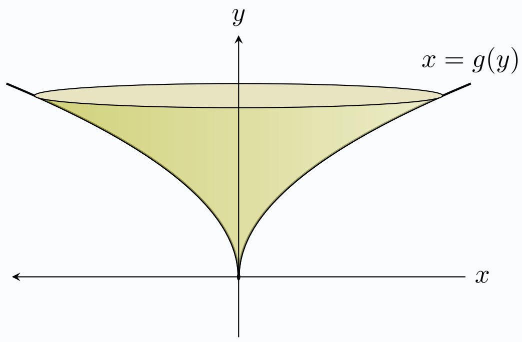
$$\begin{aligned}V &= \pi \int_1^4 x\, dx \\ &= \pi \left[\frac{x^2}{2}\right]_1^4 \\ &= \pi\left(\frac{4^2}{2} - \frac{1^2}{2}\right) \\ &= \pi\left(\frac{16}{2} - \frac{1}{2}\right) \\ &= \pi\left(8 - \frac{1}{2}\right) \\ &= \frac{15\pi}{2}\end{aligned}$$

The volume of the solid is $\frac{15\pi}{2}$ cubic units.

Proposition Volume of Revolution about the y-axis

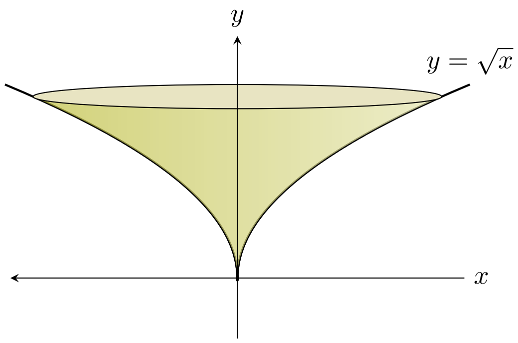
Assuming $g(y) \geq 0$ on $[c, d]$, the volume V generated by rotating the region bounded by the curve $x = g(y)$, the y-axis, and the lines $y = c$ and $y = d$ around the y-axis is given by:

$$V = \pi \int_c^d [g(y)]^2 \, dy$$



Note To use this formula, the function must be expressed in the form $x = g(y)$, where y is the independent variable. This may require rearranging the function's equation.

Ex: Find the volume of the solid generated by revolving the region bounded by the curve $y = \sqrt{x}$, the y-axis, and the line $y = 2$ about the y-axis.



Answer:

1. **Rearrange the function:** The rotation is around the y-axis, so we need to express x in terms of y:

$$y = \sqrt{x} \implies x = y^2.$$

   So, our function is $g(y) = y^2$.

2. **Identify limits:** The region is bounded by the y-axis (which corresponds to $x = 0$ and starts at $y = 0$, where the curve meets the axis) and the line $y = 2$. So, our limits are $c = 0$ and $d = 2$.

3. **Integrate:** We set up and evaluate the integral for the volume:

$$\begin{aligned} V &= \pi \int_0^2 [g(y)]^2 \, dy \\ &= \pi \int_0^2 (y^2)^2 \, dy \\ &= \pi \int_0^2 y^4 \, dy \\ &= \pi \left[\frac{y^5}{5} \right]_0^2 \\ &= \pi \left(\frac{2^5}{5} - \frac{0^5}{5} \right) \\ &= \frac{32\pi}{5} \end{aligned}$$

The volume of the solid is $\frac{32\pi}{5}$ cubic units.