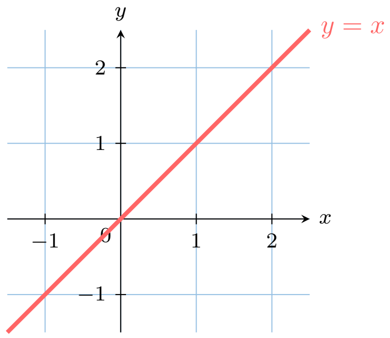
APPLICATIONS OF INTEGRATION IN GEOMETRY

A CALCULATING GEOMETRIC AREA

A.1 EVALUATING THE TOTAL GEOMETRIC AREAS USING GEOMETRIC FORMULAS

Ex 1:

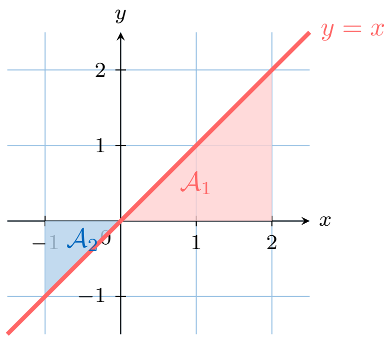


Find:

$$\int_{-1}^{2} |x|\, dx = \boxed{\frac{5}{2}}$$

Answer: The integral $\int_{-1}^{2} |x|\, dx$ corresponds to the **total geometric area** between the line $y = x$ and the x-axis. We calculate the area of the two triangles separately.

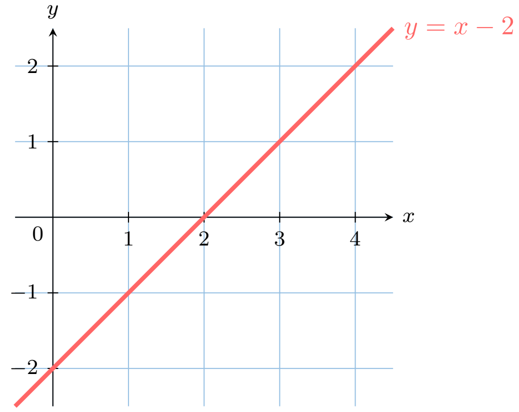
- For x from 0 to 2, the region is a triangle with base 2 and height 2.
  $\text{Area}_1 = \frac{2\times 2}{2} = 2$.
- For x from -1 to 0, the region is a triangle with base 1 and height 1 (since distance is positive).
  $\text{Area}_2 = \frac{1\times 1}{2} = 0.5$.



The total geometric area is the sum of these areas:

$$\begin{aligned}\int_{-1}^{2} |x|\, dx &= \text{Area}_1 + \text{Area}_2 \\ &= 2 + 0.5 \\ &= 2.5 \quad \text{or} \quad \frac{5}{2}.\end{aligned}$$

Ex 2:

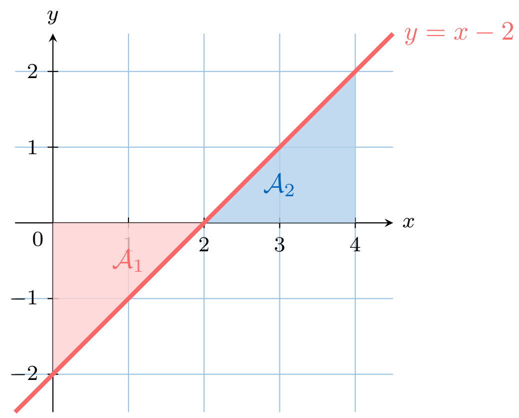


Find:

$$\int_{0}^{4} |x-2|\, dx = \boxed{4}$$

Answer: The integral $\int_0^4 |x-2|\, dx$ corresponds to the **total geometric area** between the line $y = x - 2$ and the x-axis. The line crosses the x-axis at $x = 2$.

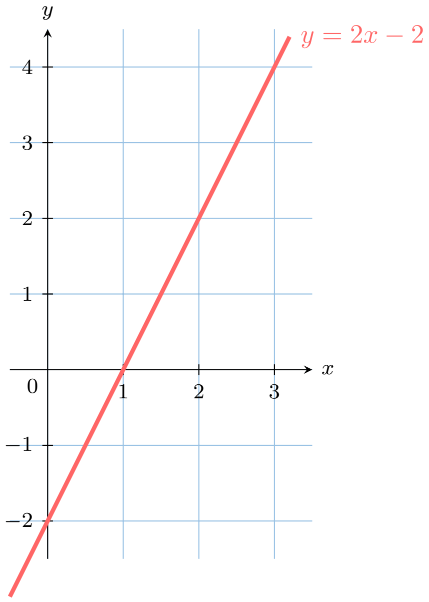
- For x from 0 to 2, the region is below the axis. It is a triangle with base 2 and height $|-2| = 2$.
  $\text{Area}_1 = \frac{2\times 2}{2} = 2$.
- For x from 2 to 4, the region is above the axis. It is a triangle with base 2 and height 2.
  $\text{Area}_2 = \frac{2\times 2}{2} = 2$.



The total geometric area is:

$$\begin{aligned}\int_{0}^{4} |x-2|\, dx &= \text{Area}_1 + \text{Area}_2 \\ &= 2 + 2 \\ &= 4.\end{aligned}$$

Ex 3:

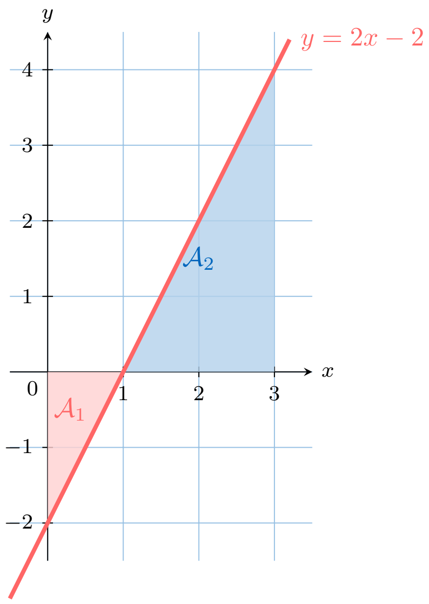


Find:

$$\int_0^3 |2x-2|\,\mathrm{d}x = \boxed{5}$$

Answer: The integral represents the total geometric area. The line $y = 2x - 2$ crosses the x-axis at $x = 1$.

- For x from 0 to 1: Triangle with base 1 and height $|-2| = 2$. $\text{Area}_1 = \frac{1\times 2}{2} = 1$.
- For x from 1 to 3: Triangle with base 2 and height $2(3)-2 = 4$. $\text{Area}_2 = \frac{2\times 4}{2} = 4$.

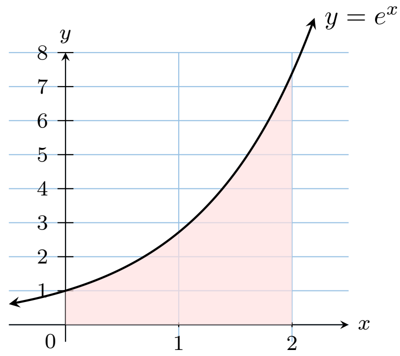


The total geometric area is:

$$\int_0^3 |2x-2|\,\mathrm{d}x = 1 + 4$$
$$= 5.$$

A.2 CALCULATING GEOMETRIC AREA

Ex 4:



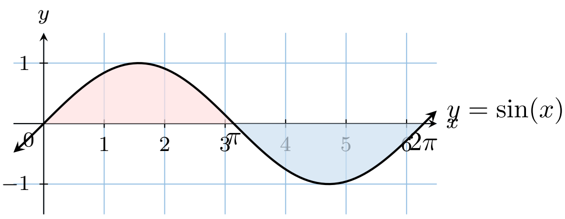
Find the area of the region enclosed by the x-axis, the curve $y = e^x$, and the lines $x = 0$ and $x = 2$.

Answer: The function $f(x) = e^x$ is always positive, so the geometric area is given directly by the definite integral.

$$\begin{aligned}\mathcal{A} &= \int_0^2 e^x\,dx \\ &= [e^x]_0^2 \\ &= e^2 - e^0 \\ &= e^2 - 1\end{aligned}$$

The area is $e^2 - 1$ square units.

Ex 5:



Find the total area of the region enclosed by the x-axis and the curve $y = \sin(x)$ from $x = 0$ to $x = 2\pi$.

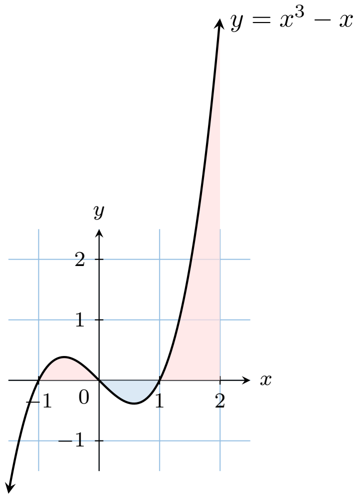
Answer: The function $y = \sin(x)$ is positive on $[0, \pi]$ and negative on $[\pi, 2\pi]$. We must calculate the area of each part separately and add their absolute values.

$$\begin{aligned}\text{Area }1(\mathcal{A}_1) &= \int_0^\pi \sin(x)dx \\ &= [-\cos(x)]_0^\pi \\ &= (-\cos(\pi)) - (-\cos(0)) \\ &= (1) - (-1) \\ &= 2 \\ \text{Area }2(\mathcal{A}_2) &= \int_\pi^{2\pi} \sin(x)dx \\ &= [-\cos(x)]_\pi^{2\pi} \\ &= (-\cos(2\pi)) - (-\cos(\pi)) \\ &= (-1) - (1) \\ &= -2\end{aligned}$$

The total geometric area is the sum of the absolute values:

$$\mathcal{A} = |\mathcal{A}_1| + |\mathcal{A}_2| = |2| + |-2| = 4$$

Ex 6:



Find the total area of the region enclosed by the x-axis and the curve $y = x^3 - x$ from $x = -1$ to $x = 2$.

Answer: The function is $f(x) = x^3 - x = x(x-1)(x+1)$. The roots are at $x = -1, 0, 1$. We must split the integral at these points.

- On $[-1, 0]$, $f(x) \geq 0$.
- On $[0, 1]$, $f(x) \leq 0$.
- On $[1, 2]$, $f(x) \geq 0$.

We calculate the integral for each sub-interval:

$$\begin{aligned}
\mathcal{A}_1 &= \int_{-1}^{0} (x^3 - x)dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} \\
&= (0) - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right) = -\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{4} \\
\mathcal{A}_2 &= \int_{0}^{1} (x^3 - x)dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1} \\
&= \left(\frac{1}{4} - \frac{1}{2}\right) - (0) = -\frac{1}{4} \\
\mathcal{A}_3 &= \int_{1}^{2} (x^3 - x)dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2} \\
&= \left(\frac{16}{4} - \frac{4}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right) = (4 - 2) - (-\frac{1}{4}) = 2 + \frac{1}{4} = \frac{9}{4}
\end{aligned}$$

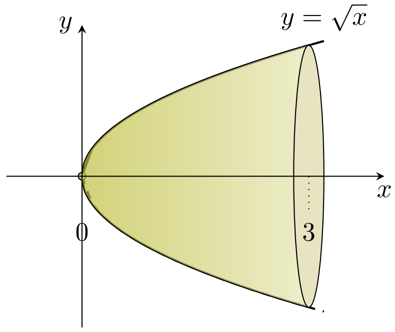
The total geometric area is the sum of the absolute values:

$$\mathcal{A} = |\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| = \left|\frac{1}{4}\right| + \left|-\frac{1}{4}\right| + \left|\frac{9}{4}\right| = \frac{1}{4} + \frac{1}{4} + \frac{9}{4} = \frac{11}{4}$$

B VOLUMES OF REVOLUTION

B.1 CALCULATING VOLUMES OF REVOLUTION ABOUT THE X-AXIS

Ex 7: Find the volume of the solid generated by revolving the region under the curve $y = \sqrt{x}$ from $x = 0$ to $x = 3$ around the x-axis.



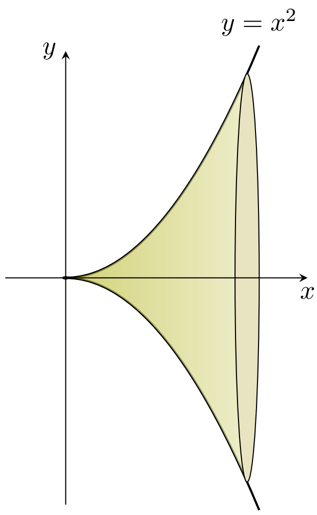
Answer:

1. **Identify:** The function is $f(x) = \sqrt{x}$, and the limits are $a = 0$ and $b = 3$.
2. **Square the function:** $[f(x)]^2 = (\sqrt{x})^2 = x$.
3. **Integrate:** We set up and evaluate the integral for the volume.

$$\begin{aligned}
V &= \pi \int_0^3 x \, dx \\
&= \pi \left[\frac{x^2}{2}\right]_0^3 \\
&= \pi \left(\frac{3^2}{2} - \frac{0^2}{2}\right) \\
&= \pi \left(\frac{9}{2} - 0\right) \\
&= \frac{9\pi}{2}
\end{aligned}$$

The volume of the solid is $\frac{9\pi}{2}$ cubic units.

Ex 8: Find the volume of the solid generated by revolving the region under the curve $y = x^2$ from $x = 0$ to $x = 3$ around the x-axis.



Answer:

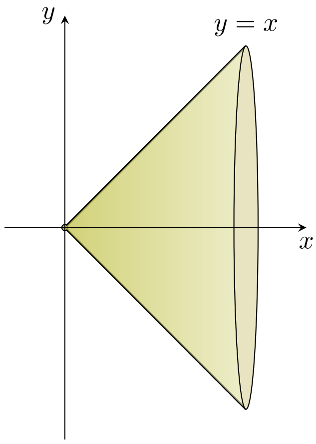
1. **Identify:** The function is $f(x) = x^2$, and the limits are $a = 0$ and $b = 3$.
2. **Square the function:** $[f(x)]^2 = \left(x^2\right)^2 = x^4$.

3. **Integrate:** We set up and evaluate the integral for the volume.

$$\begin{aligned} V &= \pi \int_0^3 x^4\,dx \\ &= \pi \left[\frac{x^5}{5}\right]_0^3 \\ &= \pi\left(\frac{3^5}{5} - \frac{0^5}{5}\right) \\ &= \pi\left(\frac{243}{5} - 0\right) \\ &= \frac{243\pi}{5} \end{aligned}$$

The volume of the solid is $\frac{243\pi}{5}$ cubic units.

Ex 9: The area bounded by the line $y = x$ and the x-axis is revolved around the x-axis to form a cone.



1. Find the volume of the cone generated if the region is from $x = 0$ to $x = 2$.

2. Find a general formula for the volume of a cone with height h and radius r by revolving the line $y = \frac{r}{h}x$ from $x = 0$ to $x = h$.

Answer:

1. For the function $f(x) = x$ on the interval $[0, 2]$, the volume of revolution is:

$$\begin{aligned} V &= \pi \int_0^2 [f(x)]^2 dx \\ &= \pi \int_0^2 x^2 dx \\ &= \pi \left[\frac{x^3}{3}\right]_0^2 \\ &= \pi\left(\frac{2^3}{3} - \frac{0^3}{3}\right) \\ &= \frac{8\pi}{3} \end{aligned}$$

This cone has a height of 2 and a radius of $f(2) = 2$. The volume is $\frac{8\pi}{3}$ cubic units.

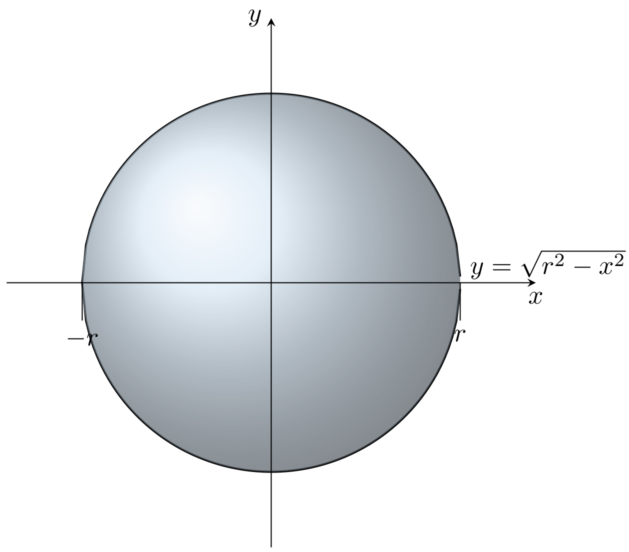
2. For the general case, the function is $f(x) = \frac{r}{h}x$. When this line is revolved around the x-axis from $x = 0$ to $x = h$, it forms a cone of height h and radius $f(h) = \frac{r}{h}h = r$.

The volume is:

$$\begin{aligned} V &= \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx \\ &= \pi \int_0^h \frac{r^2}{h^2}x^2 dx \\ &= \pi \frac{r^2}{h^2} \int_0^h x^2 dx \quad (\text{since } \frac{r^2}{h^2} \text{ is a constant}) \\ &= \pi \frac{r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h \\ &= \pi \frac{r^2}{h^2}\left(\frac{h^3}{3} - 0\right) \\ &= \pi \frac{r^2}{h^2}\frac{h^3}{3} \\ &= \frac{1}{3}\pi r^2 h \end{aligned}$$

This is the well-known formula for the volume of a cone.

Ex 10: A sphere of radius r can be generated by revolving the semi-circle $y = \sqrt{r^2 - x^2}$ from $x = -r$ to $x = r$ around the x-axis.



Use the method of volumes of revolution to prove the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$.

Answer:

1. **Identify:** The function is $f(x) = \sqrt{r^2 - x^2}$, and the limits are $a = -r$ and $b = r$.

2. **Square the function:** $[f(x)]^2 = (\sqrt{r^2 - x^2})^2 = r^2 - x^2$.

3. **Integrate:** We set up and evaluate the integral for the

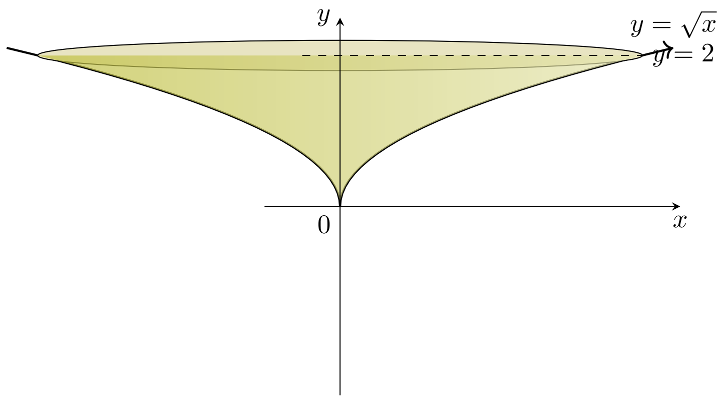
volume. Note that r is treated as a constant.

$$
\begin{aligned}
V &= \pi \int_{-r}^{r} (r^2 - x^2)\,dx \\
&= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} \\
&= \pi \left(\left(r^2(r) - \frac{r^3}{3} \right) - \left(r^2(-r) - \frac{(-r)^3}{3} \right) \right) \\
&= \pi \left(\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 - \frac{-r^3}{3} \right) \right) \\
&= \pi \left(\frac{2r^3}{3} - \left(-r^3 + \frac{r^3}{3} \right) \right) \\
&= \pi \left(\frac{2r^3}{3} - \left(-\frac{2r^3}{3} \right) \right) \\
&= \pi \left(\frac{2r^3}{3} + \frac{2r^3}{3} \right) = \pi \left(\frac{4r^3}{3} \right) \\
&= \frac{4}{3}\pi r^3
\end{aligned}
$$

This proves the well-known formula for the volume of a sphere.

B.2 CALCULATING VOLUMES OF REVOLUTION ABOUT THE Y-AXIS

Ex 11: Consider the region bounded by the curve $y = \sqrt{x}$, the y-axis, and the line $y = 2$. This region is revolved around the y-axis to generate a solid.



1. Express the boundary curve in the form $x = g(y)$.
2. Find the volume of the solid generated.

Answer:

1. **Rearrange the function:** The rotation is around the y-axis, so we must express x as a function of y.

$$y = \sqrt{x} \implies x = y^2$$

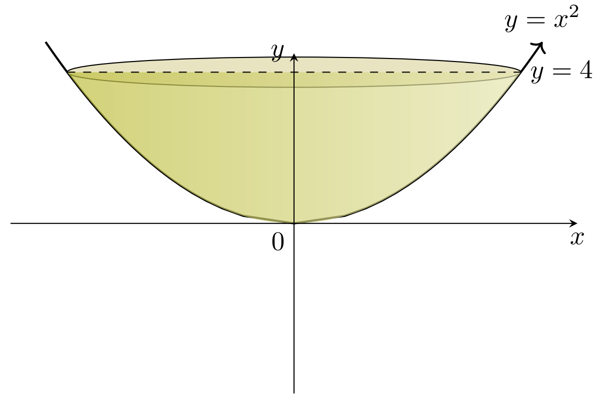
The function is $g(y) = y^2$.

2. **Find the volume:** We use the formula for revolution about the y-axis with limits from $c = 0$ to $d = 2$.

$$
\begin{aligned}
V &= \pi \int_{0}^{2} [g(y)]^2 dy \\
&= \pi \int_{0}^{2} (y^2)^2 dy \\
&= \pi \int_{0}^{2} y^4 dy \\
&= \pi \left[\frac{y^5}{5} \right]_{0}^{2} \\
&= \pi \left(\frac{2^5}{5} - \frac{0^5}{5} \right) = \frac{32\pi}{5}
\end{aligned}
$$

The volume of the solid is $\frac{32\pi}{5}$ cubic units.

Ex 12: Consider the region bounded by the curve $y = x^2$, the y-axis, and the line $y = 4$. This region is revolved around the y-axis to generate a solid.



1. Express the boundary curve in the form $x = g(y)$.
2. Find the volume of the solid generated.

Answer:

1. **Rearrange the function:** The rotation is around the y-axis, so we must express x as a function of y. Since the region is in the first quadrant ($x \geq 0$), we take the positive square root.

$$y = x^2 \implies x = \sqrt{y}$$

The function is $g(y) = \sqrt{y}$.

2. **Find the volume:** We use the formula for revolution about the y-axis with limits from $c = 0$ to $d = 4$.

$$
\begin{aligned}
V &= \pi \int_{0}^{4} [g(y)]^2 dy \\
&= \pi \int_{0}^{4} (\sqrt{y})^2 dy \\
&= \pi \int_{0}^{4} y\,dy \\
&= \pi \left[\frac{y^2}{2} \right]_{0}^{4} \\
&= \pi \left(\frac{4^2}{2} - \frac{0^2}{2} \right) = \pi \left(\frac{16}{2} \right) = 8\pi
\end{aligned}
$$

The volume of the solid is 8π cubic units.