ALGEBRAIC FRACTIONS

A DEFINITIONS

Definition **Fraction**

A **fraction** consists of two numbers: the **numerator**, a, and the **denominator**, $b \neq 0$, separated by a horizontal bar:

$a \longleftarrow$ **numerator**: number of equal parts considered

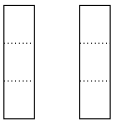
$b \longleftarrow$ **denominator**: number of equal parts the unit is divided

A fraction can be represented as:

- Symbol : $\frac{2}{3}$
- Words : two thirds or two over three
- Linear model : 

B FRACTION AS QUOTIENT

Discover: Two cakes are shared equally among three people.

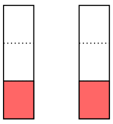


1. Use the figure to determine what fraction of the cakes each person receives.
2. Copy and complete: $\ldots$ cakes $\div \ldots$ people $= \frac{\ldots}{\ldots}$ of a cake each.

Answer:

1. Each cake is divided into three equal parts. Each person receives one piece from each cake, totaling two pieces. Since each cake is divided into three parts, each piece represents $\frac{1}{3}$ of a cake. Therefore, each person receives:

$$\frac{1}{3}+\frac{1}{3}=\frac{2}{3} \text{ of the cakes.}$$



2. 2 cakes $\div$ 3 people $= \frac{2}{3}$ of a cake each.

Proposition **Fraction as Quotient**

A fraction is a quotient that represents the result of **division**. It tells us how much of something we have when we divide it into equal parts.

- **The top number (numerator)** is the whole.
- **The bottom number (denominator)** is the number of equal parts the whole is divided into.

The fraction $\frac{a}{b}$ is the same as saying **"a divided by b"**.

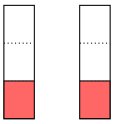
$$\frac{a}{b} = a \div b$$

The fraction $\frac{a}{b}$ is the number which, when multiplied by b, gives a:

$$\frac{a}{b} \times b = a$$

Ex:

$$2 \div 3 =$$



$$= \frac{2}{3}$$

C EQUIVALENT FRACTIONS

Definition **Equivalent Fractions**

- When you multiply the numerator and the denominator by the same number, the fractions are equals.

$$\frac{a}{b} \overset{\times k}{\underset{\times k}{=}} \frac{k\times a}{k\times b}$$

- When you divide the numerator and the denominator by the same number, the fractions are equals.

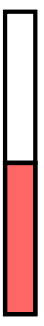
$$\frac{k\times a}{k\times b} \overset{\div k}{\underset{\div k}{=}} \frac{a}{b}$$

Ex:

$$= \frac{1}{3} \overset{\times 2}{\underset{\times 2}{=}} \frac{2\times 1}{2\times 3} = \frac{2}{6} =$$

Ex:

$$= \frac{3}{6} = \frac{3\times 1}{3\times 2} \overset{\div 3}{\underset{\div 3}{=}} \frac{1}{2} =$$



D CROSS MULTIPLICATION

Discover: We have learned that two fractions are equal if we can multiply both the numerator and the denominator by the same number.
For example:

$$\frac{2}{3} = \frac{5\times 2}{5\times 3} = \frac{10}{15}$$

Now, let's explore another way to check if two fractions are equal.
We can investigate the relationship between their numerators and denominators:

$$\begin{aligned} 2 \times 15 &= 2 \times (5 \times 3) \\ &= 5 \times 2 \times 3 \\ &= 10 \times 3 \end{aligned}$$

So, we can see that:

$$2 \times 15 = 3 \times 10$$

This leads us to a new way of checking if two fractions are equal: by cross multiplying and comparing the products.

$$\frac{2}{3} = \frac{10}{15} \quad \text{if and only if} \quad 2 \times 15 = 3 \times 10$$

This is known as the cross multiplication property.

Proposition Cross Multiplication Property

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad a \times d = b \times c$$

Ex: Solve x for $\dfrac{10}{5} = \dfrac{x}{8}$.

Answer:

$$\begin{aligned} \frac{10}{5} &= \frac{x}{8} \\ 5 \times x &= 10 \times 8 && \text{(cross mutiplication)} \\ x &= 10 \times 8 \div 5 && \text{(dividing both sides by 5)} \\ x &= 16 \end{aligned}$$

E SIMPLIFICATION

Definition Simplest form

A fraction is in **simplest form** if it is written with the smallest possible whole number numerator and denominator, that is, if its numerator and denominator have no common factors other than 1.

Ex:

- $\frac{2}{3}$ is in simplest form.
- $\frac{4}{6}$ is **not** in simplest form because we can write $\frac{4}{6} = \frac{2}{3}$.

Method Simplifying a fraction

To simplify a fraction (or to write a fraction in its simplest form), we cancel the greatest common factor of the numerator and the denominator .

Ex: Simplify $\dfrac{4}{6}$.

Answer:

$$\begin{aligned} \frac{4}{6} &= \frac{2 \times \cancel{2}}{3 \times \cancel{2}} \\ &= \frac{2}{3} \end{aligned}$$

F ADDITION AND SUBTRACTION

Definition Addition and Subtraction of Fractions with Common Denominators

- When we **add** fractions with common denominators, we keep the denominator the same and add the numerators:

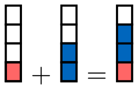
$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

- When we **subtract** fractions with common denominators, we keep the denominator the same and subtract the numerators:

$$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$$

Ex: Calculate $\frac{1}{4}+\frac{2}{4}$.

Answer:

- $$\begin{aligned}\frac{1}{4}+\frac{2}{4}&=\frac{1+2}{4}\\&=\frac{3}{4}\end{aligned}$$
- 

Method Addition or Subtraction of Fractions with Different Denominators

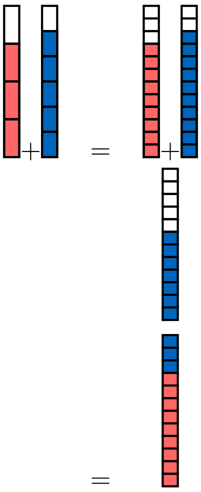
To add or subtract fractions with different denominators:

- **Find a common denominator**: Choose a common multiple of the denominators.
- **Convert each fraction**: Rewrite each fraction so it has the common denominator.
- **Add or subtract the numerators**: Add or subtract the numerators and keep the denominator the same.

Ex: Calculate $\frac{3}{4}+\frac{5}{6}$.

Answer:

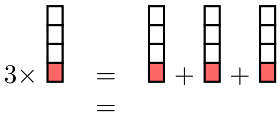
- **Find a common denominator**:To add fractions, they must have the same denominator.
  - Multiples of 4: 4, 8, **12**, 16, 20, ...
  - Multiples of 6: 6, **12**, 18, 24, ...
  - The smallest common denominator is **12**.
- $$\begin{aligned}\frac{3}{4}+\frac{5}{6}&=\frac{3\times 3}{4\times 3}+\frac{5\times 2}{6\times 2}\\&=\frac{9}{12}+\frac{10}{12}&&\text{(common denominator = 12)}\\&=\frac{9+10}{12}&&\text{(adding numerators)}\\&=\frac{19}{12}\end{aligned}$$
- **Visual representation**:



G MULTIPLICATION OF A FRACTION BY A NUMBER

Discover: Hugo has a cake. He eats $\frac{1}{4}$ of the cake each day.
How much of the cake will he have eaten after 3 days?

Answer: After 3 days, Hugo will have eaten:



$$\begin{aligned} 3 \times \frac{1}{4} &= \frac{3 \times 1}{4} \\ &= \frac{3}{4} \end{aligned}$$

So, Hugo will have eaten $\frac{3}{4}$ of the cake after 3 days.

Definition **Multiplication of a Fraction by a Number**

To multiply a fraction by a whole number:

1. Multiply the numerator by the number.
2. Keep the denominator the same.

$$a \times \frac{b}{c} = \frac{a \times b}{c}$$

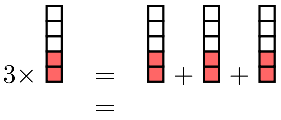
Ex: Calculate $3 \times \frac{2}{5}$.

Answer:

- Mathematical calculation:

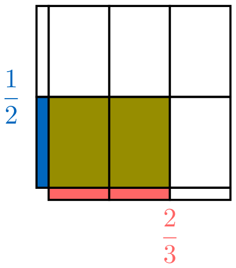
$$\begin{aligned} 3 \times \frac{2}{5} &= \frac{3 \times 2}{5} \\ &= \frac{6}{5} \end{aligned}$$

- Visual representation:



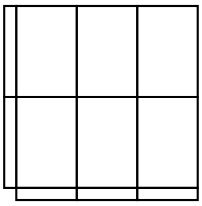
H MULTIPLICATION OF FRACTIONS

Discover: Find the area of the shaded rectangle that has sides of length $\frac{2}{3}$ and $\frac{1}{2}$.

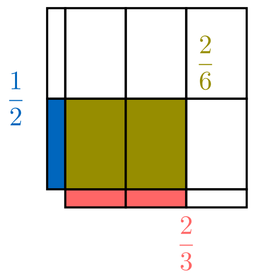


Answer:

- The unit rectangle is divided into 3 columns and 2 rows, giving a total of $3 \times 2 = 6$ equal parts.



- The shaded rectangle covers 2 columns and 1 row, so it covers $2 \times 1 = 2$ parts.



- Therefore, the area of the shaded rectangle is $\frac{2}{6}$.
- As the product of the side lengths gives the area of a rectangle, we have:

$$\begin{aligned}\frac{2}{3}\times\frac{1}{2}&=\frac{2\times 1}{3\times 2}\\&=\frac{2}{6}\end{aligned}$$

Definition Multiplication of Fractions

To multiply fractions, **tu multiplies** the numerators and **tu multiplies** the denominators:

$$\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}$$

Ex: Calculate $\frac{5}{2}\times\frac{3}{4}$.

Answer:

$$\begin{aligned}\frac{5}{2}\times\frac{3}{4}&=\frac{5\times 3}{2\times 4}\\&=\frac{15}{8}\end{aligned}$$

Method Canceling Common Factors

To make multiplication easier, **tu peux annuler** any common factors in the numerators and denominators before multiplying.

Ex: Calculate $\frac{31}{7}\times\frac{12}{31}$.

Answer:

$$\begin{aligned}\frac{31}{7}\times\frac{12}{31}&=\frac{\cancel{31}\times 12}{7\times\cancel{31}}\quad\text{(cancel the common factor 31)}\\&=\frac{12}{7}\end{aligned}$$

I DIVISION OF FRACTIONS

Definition Reciprocal

The **reciprocal** of a number is a number that, when multiplied by the original number, gives 1.

Proposition Reciprocal of a fraction

The reciprocal of the fraction $\frac{a}{b}$ is $\frac{b}{a}$.

Proof

$$\begin{aligned}\frac{a}{b}\times\frac{b}{a}&=\frac{a\times b}{b\times a}\quad\text{(les produits sont identiques)}\\&=\frac{1}{1}\\&=1\,.\end{aligned}$$

Ex: State the reciprocal of $\frac{5}{7}$.

Answer: The reciprocal of $\frac{5}{7}$ is $\frac{7}{5}$.

Definition **Division of fractions**

To divide by a fraction, you multiply by its reciprocal:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}\,,$$

or equivalently,

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c}\,.$$

Ex: Calculate $\frac{2}{3} \div \frac{5}{7}$.

Answer:

$$\begin{aligned}
\frac{2}{3} \div \frac{5}{7} &= \frac{2}{3} \times \frac{7}{5} && \text{(multiply by the reciprocal)} \\
&= \frac{2 \times 7}{3 \times 5} && \text{(multiply numerators and denominators)} \\
&= \frac{14}{15}\,.
\end{aligned}$$

J SIGN RULES

Discover: Recall from the chapter on negative numbers that dividing a positive by a negative, or a negative by a positive, yields a negative result.
Since the fraction bar represents division, consider the fraction

$$\frac{-3}{2} = \overbrace{(-3)}^{\text{negative}} \div \overbrace{2}^{\text{positive}} = \overbrace{-(3 \div 2)}^{\text{negative}} = -\frac{3}{2}\,.$$

Similarly,

$$\frac{3}{-2} = \overbrace{3}^{\text{positive}} \div \overbrace{(-2)}^{\text{negative}} = \overbrace{-(3 \div 2)}^{\text{negative}} = -\frac{3}{2}\,.$$

So, in general:

Proposition **Sign rules**

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}\,,$$

and

$$\frac{-a}{-b} = \frac{a}{b}\,.$$

Ex: Simplify $\frac{-4}{-6}$.

Answer:

$$\begin{aligned}
\frac{-4}{-6} &= \frac{4}{6} && \text{(a negative divided by a negative is positive)} \\
&= \frac{2 \times 2}{3 \times 2} && \text{(cancel the common factor 2)} \\
&= \frac{2}{3}\,.
\end{aligned}$$

K ORDER OF OPERATIONS

Definition **Order of Operations**

The division line in a fraction acts as a grouping symbol (like parentheses). This means that, according to the order of operations (PEMDAS), you must first evaluate the numerator and the denominator before performing the division.

Ex: Simplify $\frac{1+7}{3\times4}$.

Answer:

$$
\begin{aligned}
\frac{1+7}{3\times 4} &= \frac{8}{12} && \text{(evaluate numerator and denominator)}\\
&= \frac{2\times \cancel{4}}{3\times \cancel{4}} && \text{(cancel common factor)}\\
&= \frac{2}{3}
\end{aligned}
$$