ALGEBRA

A DEFINITIONS

Definition Constant

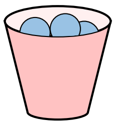
A **constant** is a number.

Ex: $0,\ 3,\ \pi$

Definition Variable

A **variable** is a quantity which we represent by a letter.

Ex:

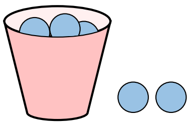


The variable x is the number of marbles inside the cup.

Definition Expression

An **expression** is an algebraic form consisting of constants, variables, and operation signs such as $+, -, \times, \div$ and $\sqrt{\ }$.

Ex:



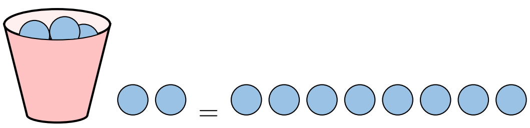
A cup contains x marbles. Next to the cup, there are 2 marbles outside. The expression for the number of marbles is

$$x+2$$

Definition Equation

An **equation** is a mathematical statement consisting of two expressions, the **left-hand side** and the **right-hand side**, separated by an equal sign $=$.

Ex:



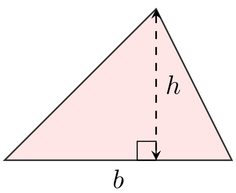
A cup contains x marbles. The equation for the number of marbles is

$$x+2=8$$

Definition Formula

A **formula** is an equation, often related to the real world, to physics or to geometry.

Ex: For a triangle:



, $A=\dfrac{b\times h}{2}$ is the formula for the area.

B NOTATIONS

Definition Product notation

We can omit the $\times$ sign when it is followed by a variable or a parenthesis.

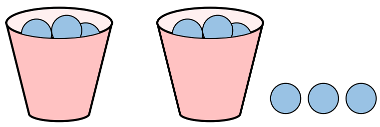
Ex:

- $2 \times x = 2x$
- $2 \times (L + l) = 2(L + l)$

Definition Repeated addition

$$\overbrace{x + x + \ldots + x}^{n \text{ terms}} = n \times x$$

Ex:



Each cup contains x marbles. Simplify the expression for the number of marbles:

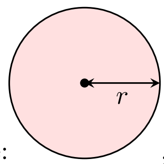
$$x + x + 1 + 1 + 1$$

Answer:

$$x + x + 1 + 1 + 1 = 2x + 3$$

Definition Repeated multiplication

$$\overbrace{x \times x \times \ldots \times x}^{n \text{ factors}} = x^n$$

Ex: For a circle: 

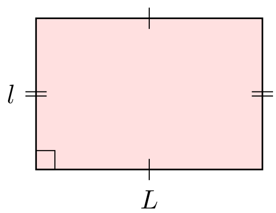
, simplify the formula for the area $A = \pi \times r \times r$.

Answer:

$$\begin{aligned} A &= \pi \times r \times r \\ &= \pi r^2 \end{aligned}$$

C IDENTITY

Discover: Three students were asked to find the formula for the perimeter of the rectangle:



They wrote:

- Su: $P = 2(l + L)$
- Louis: $P = l + L + l + L$
- Hugo: $P = 2l + 2L$

Which students are correct?

Answer: They are all correct. These three expressions $2(l+L)$, $l+L+l+L$ and $2l+2L$ produce the same result for the perimeter of the rectangle for all values of l and L. They are called identities.

Definition Identity

An **identity** is an equality between two expressions such that their evaluations produce the same value for **all** values of the variables.

Identities are fundamental in algebra: they allow us to transform and simplify expressions and are the foundation for solving equations and manipulating formulas.

Proposition Properties of Multiplying by 1 and 0

$$1 \times x = x \qquad \text{and} \qquad 0 \times x = 0$$

Proposition Commutativity Identities

$$a + b = b + a \qquad \text{and} \qquad a \times b = b \times a$$

Proposition Associativity Identities

$$(a+b)+c = a+(b+c) \qquad \text{and} \qquad (a \times b) \times c = a \times (b \times c)$$

Ex: Show that $l+L+l+L = 2l+2L$.

Answer:

$$\begin{aligned} l+L+l+L &= l+l+L+L && \text{(collecting terms)} \\ &= 2l+2L && \text{(repeated addition)} \end{aligned}$$

Method Simplifying by Collecting Like Terms

Simplifying an expression by collecting like terms involves combining terms that have the same variables raised to the same powers.

1. **Identify like terms**: Like terms are terms that have the same variable(s) raised to the same power. For example, $3x$ and $5x$ are like terms, but $3x$ and $3x^2$ are not.
2. **Combine like terms**: Add or subtract the coefficients (numerical parts) of the like terms. The variable part remains the same.

Ex: Simplify the expression: $2x+4+x-2$

Answer:

$$\begin{aligned} 2x+4+x-2 &= 2x+4+x-2 && \text{(identifying like terms)} \\ &= (2+1)x+4-2 && \text{(combining like terms)} \\ &= 3x+2 && \text{(simplifying)} \end{aligned}$$

D SUBSTITUTING

Definition Substituting

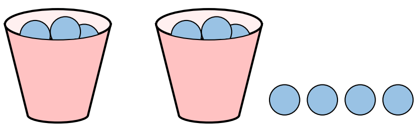
Substituting is replacing a variable in an expression or equation with a specific value.

To avoid confusion with signs, especially when substituting negative values, we usually write substitutions in parentheses.

Method Evaluating

To **evaluate** an expression, substitute a number for each variable and perform the arithmetic operations.

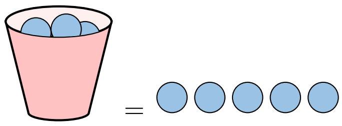
Ex:



Each cup contains x marbles. The expression for the total number of marbles is

$$2x + 4$$

Evaluate this expression when $x = 5$ (meaning there are 5 marbles in each cup):



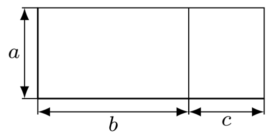
Answer:

$$\begin{aligned} 2x + 4 &= 2 \times (5) + 4 \quad \text{(substituting } x \text{ by } 5\text{)} \\ &= 10 + 4 \\ &= 14 \end{aligned}$$

There are 14 marbles.

E DISTRIBUTIVE IDENTITIES

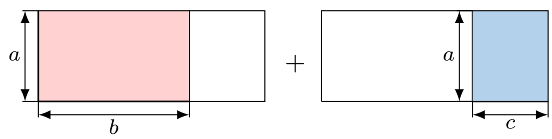
Discover: The large rectangle below is split into two smaller rectangles. Find the total area of the large rectangle in two different ways.



Answer:

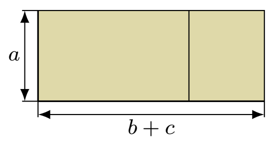
- **Method 1: Sum of the parts**
  The total area is the sum of the areas of the two smaller rectangles.

$$\text{Total Area} = \text{Area 1} + \text{Area 2} = ab + ac$$



- **Method 2: Area of the whole**
  The total length of the base is $b + c$ and the height is a.

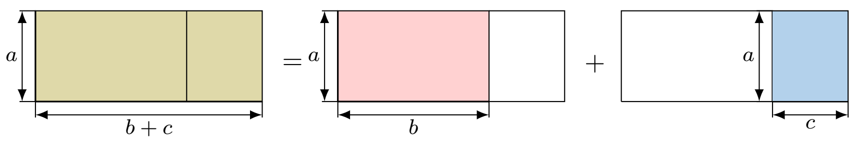
$$\text{Total Area} = a(b + c)$$



Since both methods calculate the same total area, the two expressions must be equal. This gives us the identity:

$$a(b + c) = ab + ac$$

This important rule is known as the **distributive law**.



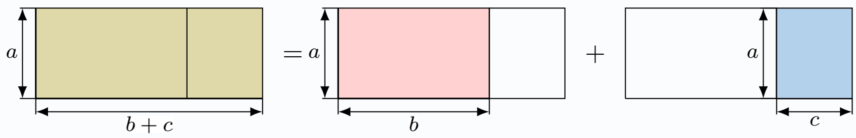
$a(b + c) \quad = \quad ab \quad + \quad ac$

Proposition Distributive Law

Multiplication is distributive over addition and subtraction:

- **Addition:**

$$a\,(b+c) = ab + ac$$



- **Subtraction:**

$$a\,(b-c) = ab - ac$$

Ex: Show that $2(\ell + L) = 2\ell + 2L$.

Answer:

$$\begin{aligned} 2\,(\ell + L) &= 2\times\ell + 2\times L \\ &= 2\ell + 2L \end{aligned}$$

So $2(\ell + L) = 2\ell + 2L$.

Definition Expanding

Expanding is the process of using the distributive law to write a product with parentheses as a sum (or difference) of terms.

Ex: Expand $2(2x + 3)$.

Answer:

$$\begin{aligned} 2\,(2x + 3) &= 2\times 2x + 2\times 3 \\ &= 4x + 6 \end{aligned}$$

So $2(2x + 3) = 4x + 6$.