AFFINE TRANSFORMATIONS

A LINEAR TRANSFORMATIONS

A.1 APPLYING LINEAR TRANSFORMATIONS

Ex 1: Find the image of the point $P(3, -2)$ under the transformation defined by the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$.

$$P'(\boxed{8}, \boxed{-5})$$

Answer: Let $\mathbf{p} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ be the position vector of P.

$$\begin{aligned} \mathbf{p}' &= \mathbf{A}\mathbf{p} \\ &= \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 2(3) + (-1)(-2) \\ 1(3) + 4(-2) \end{pmatrix} \\ &= \begin{pmatrix} 6 + 2 \\ 3 - 8 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ -5 \end{pmatrix} \end{aligned}$$

The image is $P'(8, -5)$.

Ex 2: Find the image of the point $Q(4, 1)$ under the transformation defined by the matrix $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$.

$$Q'(\boxed{-2}, \boxed{12})$$

Answer: Let $\mathbf{q} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ be the position vector of Q.

$$\begin{aligned} \mathbf{q}' &= \mathbf{B}\mathbf{q} \\ &= \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1(4) + 2(1) \\ 3(4) + 0(1) \end{pmatrix} \\ &= \begin{pmatrix} -4 + 2 \\ 12 + 0 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 12 \end{pmatrix} \end{aligned}$$

The image is $Q'(-2, 12)$.

Ex 3: Find the image of the point $R(-2, 5)$ under the transformation defined by the matrix $\mathbf{M} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}$.

$$R'(\boxed{13}, \boxed{14})$$

Answer: Let $\mathbf{r} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ be the position vector of R.

$$\begin{aligned} \mathbf{r}' &= \mathbf{M}\mathbf{r} \\ &= \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 1(-2) + 3(5) \\ -2(-2) + 2(5) \end{pmatrix} \\ &= \begin{pmatrix} -2 + 15 \\ 4 + 10 \end{pmatrix} \\ &= \begin{pmatrix} 13 \\ 14 \end{pmatrix} \end{aligned}$$

The image is $R'(13, 14)$.

Ex 4: Find the image of the point $S(2, 3)$ under the transformation defined by the matrix $\mathbf{C} = \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}$.

$$S'(\boxed{2}, \boxed{11})$$

Answer: Let $\mathbf{s} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ be the position vector of S.

$$\begin{aligned} \mathbf{s}' &= \mathbf{C}\mathbf{s} \\ &= \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 4(2) + (-2)(3) \\ 1(2) + 3(3) \end{pmatrix} \\ &= \begin{pmatrix} 8 - 6 \\ 2 + 9 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 11 \end{pmatrix} \end{aligned}$$

The image is $S'(2, 11)$.

B STANDARD LINEAR TRANSFORMATIONS

B.1 FINDING THE MATRIX REPRESENTATION

Ex 5: Find the matrix representing a vertical stretch by a factor of 4.

$$\mathbf{S} = \begin{pmatrix} \boxed{1} & \boxed{0} \\ \boxed{0} & \boxed{4} \end{pmatrix}$$

Answer: A vertical stretch by factor k corresponds to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$.
Here, $k = 4$, so the matrix is:

$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

Ex 6: Find the matrix representing a horizontal stretch by a factor of 3.

$$\mathbf{S} = \begin{pmatrix} \boxed{3} & \boxed{0} \\ \boxed{0} & \boxed{1} \end{pmatrix}$$

Answer: A horizontal stretch by factor k corresponds to the matrix $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$.
Here, $k = 3$, so the matrix is:

$$\mathbf{S} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

Ex 7: Find the matrix representing a reflection in the x-axis.

$$\mathbf{M} = \begin{pmatrix} \boxed{1} & \boxed{0} \\ \boxed{0} & \boxed{-1} \end{pmatrix}$$

Answer: A reflection in the x-axis maps $(1,0) \to (1,0)$ and $(0,1) \to (0,-1)$.
The matrix is:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Ex 8: Find the matrix representing a reflection in the y-axis.

$$\mathbf{M} = \begin{pmatrix} \boxed{-1} & \boxed{0} \\ \boxed{0} & \boxed{1} \end{pmatrix}$$

Answer: A reflection in the y-axis maps $(1,0) \to (-1,0)$ and $(0,1) \to (0,1)$.
The matrix is:

$$\mathbf{M} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Ex 9: Find the matrix representing a reflection in the line $y = x$.

$$\mathbf{M} = \begin{pmatrix} \boxed{0} & \boxed{1} \\ \boxed{1} & \boxed{0} \end{pmatrix}$$

Answer: A reflection in the line $y = x$ swaps the coordinates: $(1,0) \to (0,1)$ and $(0,1) \to (1,0)$.
The matrix is:

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Ex 10: Find the matrix representing an enlargement (homothety) centered at the origin with a scale factor of 2.

$$\mathbf{H} = \begin{pmatrix} \boxed{2} & \boxed{0} \\ \boxed{0} & \boxed{2} \end{pmatrix}$$

Answer: An enlargement centered at the origin with scale factor k corresponds to the matrix $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.
Here, $k = 2$, so the matrix is:

$$\mathbf{H} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Ex 11: Find the matrix representing a rotation of $\frac{\pi}{2}$ radians anti-clockwise about the origin.

$$\mathbf{R} = \begin{pmatrix} \boxed{0} & \boxed{-1} \\ \boxed{1} & \boxed{0} \end{pmatrix}$$

Answer: A rotation of angle θ anti-clockwise about the origin is represented by the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.
Here, $\theta = \frac{\pi}{2}$. Since $\cos(\frac{\pi}{2}) = 0$ and $\sin(\frac{\pi}{2}) = 1$, the matrix is:

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

MCQ 12: The transformation T is a reflection in the line $y = x\sqrt{3}$.
Find the matrix $\mathbf{T}$ that represents this transformation.

☐ $\mathbf{T} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

☒ $\mathbf{T} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

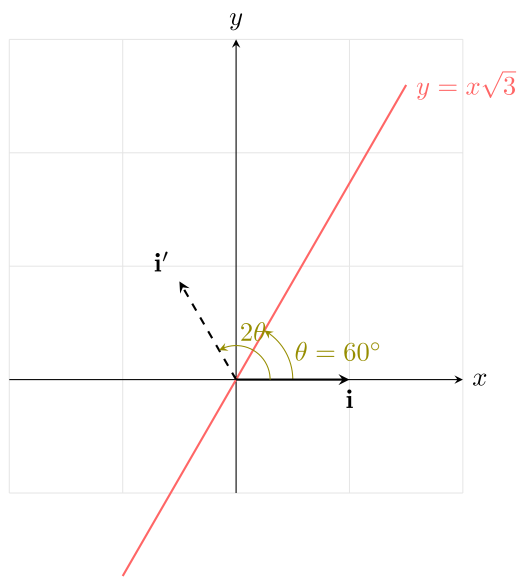
☐ $\mathbf{T} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

☐ $\mathbf{T} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

Answer: The matrix for a reflection in the line $y = (\tan\theta)x$ is given by:

$$\mathbf{T} = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

1. Identify the angle θ: The equation is $y = x\sqrt{3}$. The gradient is $\tan\theta = \sqrt{3}$.
$$\theta = \arctan(\sqrt{3}) = 60^\circ \quad (\text{or } \frac{\pi}{3} \text{ radians}).$$
2. Calculate 2θ:
$$2\theta = 120^\circ \quad (\text{or } \frac{2\pi}{3} \text{ radians}).$$
3. Substitute into the matrix:
$$\mathbf{T} = \begin{pmatrix} \cos(120^\circ) & \sin(120^\circ) \\ \sin(120^\circ) & -\cos(120^\circ) \end{pmatrix}$$
Since $\cos(120^\circ) = -\frac{1}{2}$ and $\sin(120^\circ) = \frac{\sqrt{3}}{2}$:
$$\mathbf{T} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$



MCQ 13: The transformation T is a reflection in the line $y = -x\sqrt{3}$.
Find the matrix $\mathbf{T}$ that represents this transformation.

- ☐ $\mathbf{T} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$
- ☐ $\mathbf{T} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
- ☒ $\mathbf{T} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$
- ☐ $\mathbf{T} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

Answer: The matrix for a reflection in the line $y = (\tan\theta)x$ is given by:

$$\mathbf{T} = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

1. Identify the angle θ: The equation is $y = -x\sqrt{3}$. The gradient is $\tan\theta = -\sqrt{3}$.

$$\theta = \arctan(-\sqrt{3}) = -60^\circ \quad (\text{or } -\frac{\pi}{3} \text{ radians}).$$

2. Calculate 2θ:

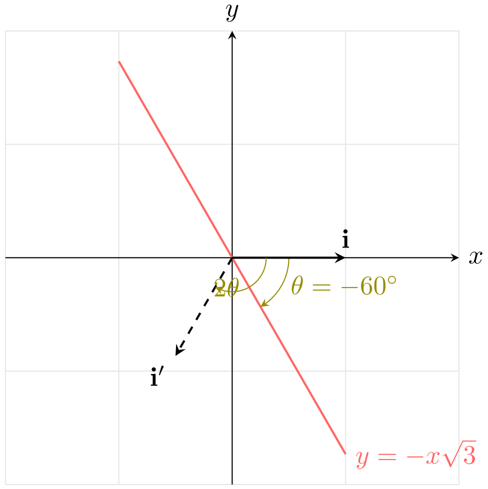
$$2\theta = -120^\circ \quad (\text{or } -\frac{2\pi}{3} \text{ radians}).$$

3. Substitute into the matrix:

$$\mathbf{T} = \begin{pmatrix} \cos(-120^\circ) & \sin(-120^\circ) \\ \sin(-120^\circ) & -\cos(-120^\circ) \end{pmatrix}$$

Since $\cos(-120^\circ) = -\frac{1}{2}$ and $\sin(-120^\circ) = -\frac{\sqrt{3}}{2}$:

$$\mathbf{T} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -(-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$



B.2 APPLYING STANDARD LINEAR TRANSFORMATIONS

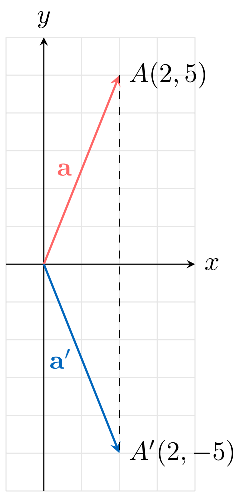
Ex 14: Find the image of the point $A(2,5)$ under a reflection in the x-axis.

$A'(\boxed{2}, \boxed{-5})$

Answer: The matrix for reflection in the x-axis is $\mathbf{M}_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$\begin{aligned} \mathbf{a}' &= \mathbf{M}_x \mathbf{a} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -5 \end{pmatrix} \end{aligned}$$

Alternatively, recall that reflection in the x-axis maps $(x,y) \to (x,-y)$, so $(2,5) \to (2,-5)$.



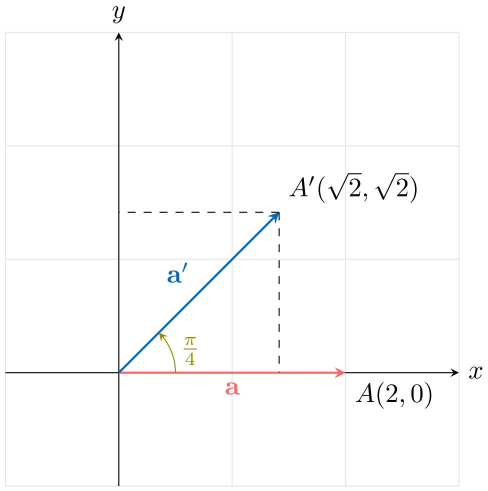
Ex 15: Find the image of the point $A(2,0)$ under a rotation of $\frac{\pi}{4}$ radians (45°) anti-clockwise about the origin.

$A'(\boxed{\sqrt{2}}, \boxed{\sqrt{2}})$

Answer: The matrix for a rotation of angle θ is $\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.
For $\theta = \frac{\pi}{4}$, we have $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

$$\begin{aligned} \mathbf{a}' &= \mathbf{R}_{\frac{\pi}{4}} \mathbf{a} \\ &= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\sqrt{2}}{2}(2) - 0 \\ \frac{\sqrt{2}}{2}(2) + 0 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} \end{aligned}$$

The image is $A'(\sqrt{2}, \sqrt{2})$.



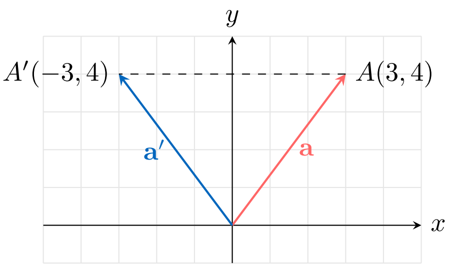
Ex 16: Find the image of the point $A(3,4)$ under a reflection in the y-axis.

$$A'(\boxed{-3}, \boxed{4})$$

Answer: The matrix for reflection in the y-axis is $\mathbf{M}_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\begin{aligned} \mathbf{a}' &= \mathbf{M}_y \mathbf{a} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 4 \end{pmatrix} \end{aligned}$$

Alternatively, recall that reflection in the y-axis maps $(x, y) \to (-x, y)$, so $(3, 4) \to (-3, 4)$.

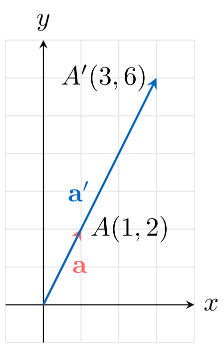


Ex 17: Find the image of the point $A(1,2)$ under an enlargement centered at the origin with a scale factor of 3.

$$A'(\boxed{3}, \boxed{6})$$

Answer: The matrix for an enlargement with scale factor $k = 3$ is $\mathbf{S} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

$$\begin{aligned} \mathbf{a}' &= \mathbf{S}\mathbf{a} \\ &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 6 \end{pmatrix} \end{aligned}$$



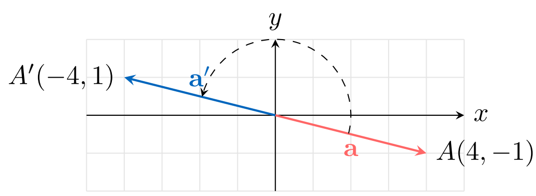
Ex 18: Find the image of the point $A(4,-1)$ under a rotation of π radians $(180°)$ about the origin.

$$A'(\boxed{-4}, \boxed{1})$$

Answer: The matrix for rotation of π is $\mathbf{R}_\pi = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

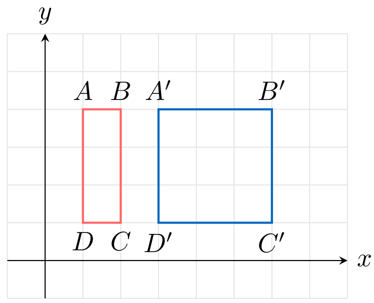
$$\begin{aligned} \mathbf{a}' &= \mathbf{R}_\pi \mathbf{a} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 1 \end{pmatrix} \end{aligned}$$

This corresponds to a point reflection in the origin: $(x, y) \to (-x, -y)$.



B.3 IDENTIFYING THE TRANSFORMATION

Ex 19: A linear transformation maps rectangle $ABCD$ onto square $A'B'C'D'$.

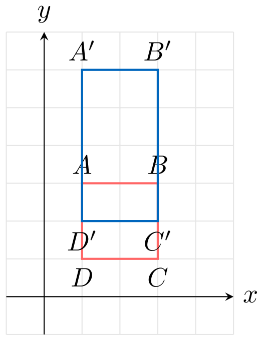


1. Identify the transformation.
2. Write down the transformation equation in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Answer:

1. **Identify the transformation:**
   - The rectangle $ABCD$ has width $2 - 1 = 1$ and height $4 - 1 = 3$.
   - The square $A'B'C'D'$ has width $6 - 3 = 3$ and height $4 - 1 = 3$.
   - The height remains unchanged, but the width is multiplied by 3. This indicates a horizontal stretch by scale factor $k = 3$.
   - After stretching horizontally by factor 3, point $D(1, 1)$ would move to $(3, 1)$, which matches D'.
   - Therefore, the transformation is a **horizontal stretch with scale factor 3**.
2. Since it is a horizontal stretch by factor 3, the matrix is $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. The equation is $\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}$.

Ex 20: A linear transformation maps square $ABCD$ onto rectangle $A'B'C'D'$.

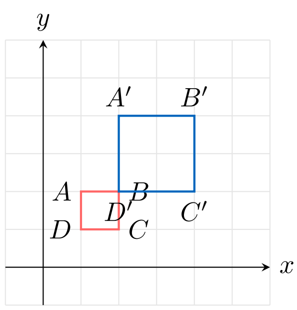


1. Identify the transformation.
2. Write down the transformation equation in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Answer:

1. **Identify the transformation:**
   - The square $ABCD$ has width $3-1=2$ and height $3-1=2$.
   - The rectangle $A'B'C'D'$ has width $3-1=2$ and height $6-2=4$.
   - The width remains unchanged, but the height is multiplied by 2. This indicates a vertical stretch by scale factor $k=2$.
   - Check point $D(1,1)$: A vertical stretch by 2 should map $(x,y) \to (x,2y)$. $D(1,1) \to (1,2)$, which matches D'.
   - Therefore, the transformation is a **vertical stretch with scale factor 2**.
2. Since it is a vertical stretch by factor 2, the matrix is $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. The equation is $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x}$.

Ex 21: A linear transformation maps square $ABCD$ onto square $A'B'C'D'$.

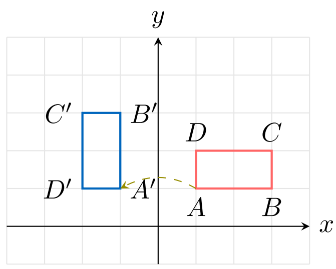


1. Identify the transformation.
2. Write down the transformation equation in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Answer:

1. **Identify the transformation:**
   - The square $ABCD$ has side length 1.
   - The square $A'B'C'D'$ has side length 2.
   - Both dimensions are multiplied by 2. This suggests an enlargement (homothety).
   - Check point $C(2,1)$: An enlargement of scale factor 2 centered at the origin maps $(x,y) \to (2x,2y)$. $C(2,1) \to (4,2)$, which matches C'.
   - Therefore, the transformation is an **enlargement with scale factor 2 centered at the origin**.
2. Since it is an enlargement by factor 2, the matrix is $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. The equation is $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x}$.

Ex 22: A linear transformation maps rectangle $ABCD$ onto rectangle $A'B'C'D'$.



1. Identify the transformation.
2. Write down the transformation equation in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Answer:

1. **Identify the transformation:**
   - The shape has been rotated.
   - Let's check point $B(3,1)$. Its image is $B'(-1,3)$.
   - The distance from the origin is preserved ($\sqrt{3^2+1^2} = \sqrt{(-1)^2+3^2}$).
   - The angle of B is $\arctan(1/3) \approx 18.4°$. The angle of B' is $180° - \arctan(3) \approx 108.4°$. The difference is $90°$.
   - We can verify with $A(1,1) \to A'(-1,1)$. This is clearly a $90°$ turn to the left.
   - Therefore, the transformation is a **rotation of $\frac{\pi}{2}$ (90°) anti-clockwise about the origin**.
2. The matrix for a rotation of $\frac{\pi}{2}$ is $\mathbf{A} = \begin{pmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
   The equation is $\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x}$.

C TRANSLATION

C.1 APPLYING TRANSLATIONS

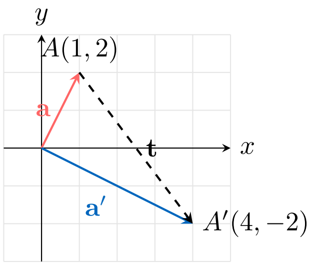
Ex 23: Find the image of the point $A(1,2)$ under a translation by the vector $\mathbf{t} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

$$A'(\boxed{4}, \boxed{-2})$$

Answer: A translation maps a point A to A' by adding the translation vector to the position vector: $\mathbf{a}' = \mathbf{a} + \mathbf{t}$.

$$\begin{aligned}\mathbf{a}' &= \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}3\\-4\end{pmatrix}\\ &= \begin{pmatrix}1+3\\2-4\end{pmatrix}\\ &= \begin{pmatrix}4\\-2\end{pmatrix}\end{aligned}$$

The image is $A'(4,-2)$.



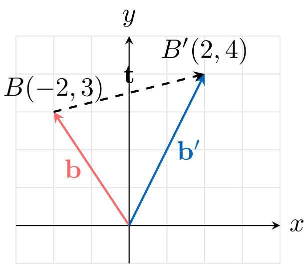
Ex 24: Find the image of the point $B(-2,3)$ under a translation by the vector $\mathbf{t} = \begin{pmatrix}4\\1\end{pmatrix}$.

$B'(\boxed{2}, \boxed{4})$

Answer: A translation maps a point B to B' by adding the translation vector to the position vector: $\mathbf{b}' = \mathbf{b} + \mathbf{t}$.

$$\begin{aligned}\mathbf{b}' &= \begin{pmatrix}-2\\3\end{pmatrix} + \begin{pmatrix}4\\1\end{pmatrix}\\ &= \begin{pmatrix}-2+4\\3+1\end{pmatrix}\\ &= \begin{pmatrix}2\\4\end{pmatrix}\end{aligned}$$

The image is $B'(2,4)$.



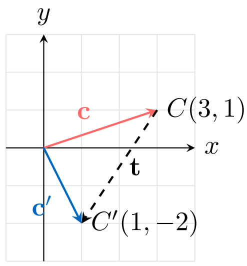
Ex 25: Find the image of the point $C(3,1)$ under a translation by the vector $\mathbf{t} = \begin{pmatrix}-2\\-3\end{pmatrix}$.

$C'(\boxed{1}, \boxed{-2})$

Answer: A translation maps a point C to C' by adding the translation vector to the position vector: $\mathbf{c}' = \mathbf{c} + \mathbf{t}$.

$$\begin{aligned}\mathbf{c}' &= \begin{pmatrix}3\\1\end{pmatrix} + \begin{pmatrix}-2\\-3\end{pmatrix}\\ &= \begin{pmatrix}3-2\\1-3\end{pmatrix}\\ &= \begin{pmatrix}1\\-2\end{pmatrix}\end{aligned}$$

The image is $C'(1,-2)$.



D AFFINE TRANSFORMATION

D.1 APPLYING AN AFFINE TRANSFORMATION

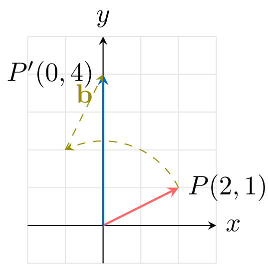
Ex 26: Find the image of the point $P(2,1)$ under the affine transformation defined by $\mathbf{x}' = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}1\\2\end{pmatrix}$.

$P'(\boxed{0}, \boxed{4})$

Answer: We apply the linear transformation (rotation) first, then add the translation vector.

$$\begin{aligned}\mathbf{p}' &= \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix}\\ &= \begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix}\\ &= \begin{pmatrix}0\\4\end{pmatrix}\end{aligned}$$

The image is $P'(0,4)$.



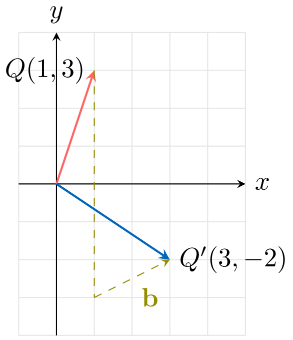
Ex 27: Find the image of the point $Q(1,3)$ under the affine transformation defined by $\mathbf{x}' = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{x} + \begin{pmatrix}2\\1\end{pmatrix}$.

$Q'(\boxed{3}, \boxed{-2})$

Answer: We apply the linear transformation (reflection in the x-axis) first, then add the translation vector.

$$\begin{aligned}\mathbf{q}' &= \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix}\\ &= \begin{pmatrix}1\\-3\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix}\\ &= \begin{pmatrix}3\\-2\end{pmatrix}\end{aligned}$$

The image is $Q'(3,-2)$.



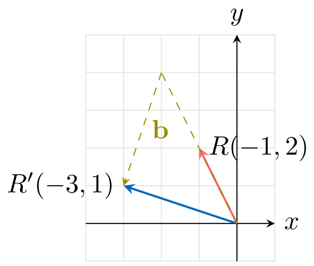
Ex 28: Find the image of the point $R(-1, 2)$ under the affine transformation defined by $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -1 \\ -3 \end{pmatrix}$.

$$R'(\boxed{-3}, \boxed{1})$$

Answer: We apply the linear transformation (enlargement scale factor 2) first, then add the translation vector.

$$\begin{aligned} \mathbf{r}' &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 1 \end{pmatrix} \end{aligned}$$

The image is $R'(-3, 1)$.



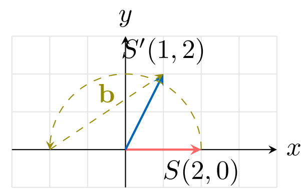
Ex 29: Find the image of the point $S(2, 0)$ under the affine transformation defined by $\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

$$S'(\boxed{1}, \boxed{2})$$

Answer: We apply the linear transformation (rotation of 180°) first, then add the translation vector.

$$\begin{aligned} \mathbf{s}' &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

The image is $S'(1, 2)$.



D.2 IDENTIFYING THE TRANSFORMATION

Ex 30: An affine transformation is defined by $\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -2 \\ 5 \end{pmatrix}$.

1. Find the image of the origin $O(0, 0)$.
2. Find the image of the point $A(1, 4)$.
3. Describe the transformation geometrically.

Answer:

1. **Image of Origin:**

$$\mathbf{o}' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}.$$

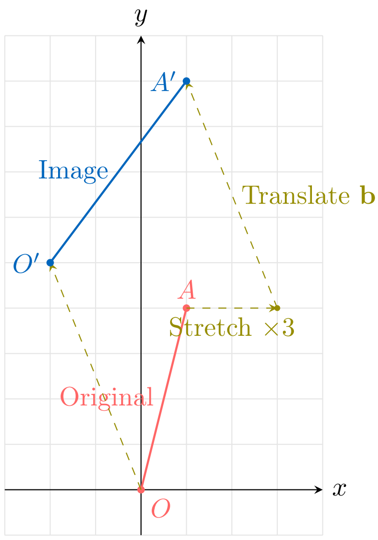
Image is $(-2, 5)$.

2. **Image of A:**

$$\mathbf{a}' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}.$$

Image is $(1, 9)$.

3. **Description:** It is a horizontal stretch by scale factor 3 followed by a translation by vector $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$.



Ex 31: An affine transformation is defined by $\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

1. Find the image of the origin $O(0, 0)$.
2. Find the image of the point $A(2, 0)$.
3. Describe the transformation geometrically.

Answer:

1. **Image of Origin:**

$$\mathbf{o}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

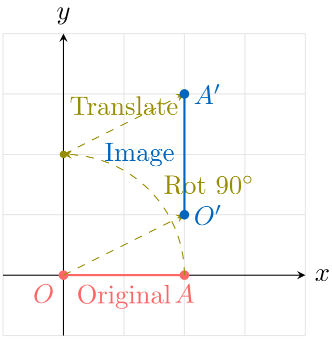
Image is $(2, 1)$.

2. **Image of A:**

$$\begin{aligned}\mathbf{a}' &= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}\\ &= \begin{pmatrix}0\\2\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}\\ &= \begin{pmatrix}2\\3\end{pmatrix}.\end{aligned}$$

Image is $(2,3)$.

3. **Description:** It is a rotation of $90°$ $(\frac{\pi}{2})$ anti-clockwise about the origin, followed by a translation by vector $\begin{pmatrix}2\\1\end{pmatrix}$.



E COMPOSITION OF TRANSFORMATIONS

E.1 FINDING THE MATRIX REPRESENTATION

Ex 32: Find the single matrix representing a reflection in the y-axis followed by a horizontal stretch of scale factor 3.

$$\mathbf{T} = \begin{pmatrix}\boxed{-3} & \boxed{0}\\ \boxed{0} & \boxed{1}\end{pmatrix}$$

Answer: Reflection in y-axis: $\mathbf{M}_y = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$.

Horizontal stretch (factor 3): $\mathbf{S} = \begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}$.

The combined transformation is $\mathbf{T} = \mathbf{S}\mathbf{M}_y$ (stretch applied **after** reflection).

$$\begin{aligned}\mathbf{T} &= \begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}3(-1)+0(0) & 3(0)+0(1)\\ 0(-1)+1(0) & 0(0)+1(1)\end{pmatrix}\\ &= \begin{pmatrix}-3 & 0\\ 0 & 1\end{pmatrix}\end{aligned}$$

Ex 33: Find the single matrix representing a rotation of $90°$ anti-clockwise followed by a vertical stretch of scale factor 2.

$$\mathbf{T} = \begin{pmatrix}\boxed{0} & \boxed{-1}\\ \boxed{2} & \boxed{0}\end{pmatrix}$$

Answer: Rotation of $90°$: $\mathbf{R}_{90} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$.

Vertical stretch (factor 2): $\mathbf{S} = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$.

The combined transformation is $\mathbf{T} = \mathbf{S}\mathbf{R}_{90}$ (stretch applied **after** rotation).

$$\begin{aligned}\mathbf{T} &= \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\\ &= \begin{pmatrix}1(0)+0(1) & 1(-1)+0(0)\\ 0(0)+2(1) & 0(-1)+2(0)\end{pmatrix}\\ &= \begin{pmatrix}0 & -1\\ 2 & 0\end{pmatrix}\end{aligned}$$

Ex 34: Find the single matrix representing a horizontal stretch of scale factor 4 followed by a reflection in the x-axis.

$$\mathbf{T} = \begin{pmatrix}\boxed{4} & \boxed{0}\\ \boxed{0} & \boxed{-1}\end{pmatrix}$$

Answer: Horizontal stretch (factor 4): $\mathbf{S} = \begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}$.

Reflection in x-axis: $\mathbf{M}_x = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

The combined transformation is $\mathbf{T} = \mathbf{M}_x\mathbf{S}$ (reflection applied **after** stretch).

$$\begin{aligned}\mathbf{T} &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}1(4)+0(0) & 1(0)+0(1)\\ 0(4)+(-1)(0) & 0(0)+(-1)(1)\end{pmatrix}\\ &= \begin{pmatrix}4 & 0\\ 0 & -1\end{pmatrix}\end{aligned}$$

E.2 FINDING THE COMPOSITE MATRIX

Ex 35: Let T_1 be a rotation of $90°$ anti-clockwise and T_2 be a reflection in the line $y = x$.

1. Find the matrix $\mathbf{A}$ for T_1 and $\mathbf{B}$ for T_2.
2. Calculate the matrix for the composite transformation $T_2 \circ T_1$ (T_1 followed by T_2).
3. Identify the resulting transformation.

Answer:

1. Matrices:

$$\mathbf{A} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

2. The composite matrix is $\mathbf{C} = \mathbf{B}\mathbf{A}$.

$$\begin{aligned}\mathbf{C} &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\\ &= \begin{pmatrix}0(0)+1(1) & 0(-1)+1(0)\\ 1(0)+0(1) & 1(-1)+0(0)\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\end{aligned}$$

3. The resulting matrix $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ represents a reflection in the x-axis.

Ex 36: Let T_1 be a reflection in the x-axis and T_2 be a reflection in the y-axis.

1. Find the matrix $\mathbf{A}$ for T_1 and $\mathbf{B}$ for T_2.
2. Calculate the matrix for the composite transformation $T_2 \circ T_1$ (T_1 followed by T_2).
3. Identify the resulting transformation.

Answer:

1. Matrices:
$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
2. The composite matrix is $\mathbf{C} = \mathbf{BA}$.
$$\begin{aligned} \mathbf{C} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1(1)+0(0) & -1(0)+0(-1) \\ 0(1)+1(0) & 0(0)+1(-1) \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$
3. The resulting matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ represents a rotation of 180° (π radians) about the origin (or a point reflection in the origin).

Ex 37: Let T_1 be a reflection in the line $y = x$ and T_2 be a reflection in the x-axis.

1. Find the matrix $\mathbf{A}$ for T_1 and $\mathbf{B}$ for T_2.
2. Calculate the matrix for the composite transformation $T_2 \circ T_1$ (T_1 followed by T_2).
3. Identify the resulting transformation.

Answer:

1. Matrices:
$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
2. The composite matrix is $\mathbf{C} = \mathbf{BA}$.
$$\begin{aligned} \mathbf{C} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1(0)+0(1) & 1(1)+0(0) \\ 0(0)+(-1)(1) & 0(1)+(-1)(0) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$
3. The resulting matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ represents a rotation of 90° clockwise (or 270° anti-clockwise) about the origin.

E.3 CALCULATING AND INVERTING COMPOSITE MATRICES

Ex 38: Consider two transformations:

- T_1: A reflection in the line $y = x\sqrt{3}$.
- T_2: An enlargement centered at the origin with scale factor 2.

1. Find the matrix $\mathbf{A}$ representing T_1. (Recall: $\tan(60^\circ) = \sqrt{3}$).
2. Find the matrix $\mathbf{B}$ representing T_2.
3. Find the matrix $\mathbf{C}$ representing the composite transformation T_1 followed by T_2.
4. Find the image of the point $Q(2, 0)$ under this composite transformation.
5. Find the coordinates of point R such that its image under this composite transformation is $R'(2, 2\sqrt{3})$.
(You are given that $\mathbf{C}^{-1} = \begin{pmatrix} -0.25 & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & 0.25 \end{pmatrix}$).

Answer:

1. **Matrix A:**
The line $y = x\sqrt{3}$ has an angle $\theta = 60^\circ$.
Reflection matrix $\mathbf{A} = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$.
With $2\theta = 120^\circ$:
$$\mathbf{A} = \begin{pmatrix} \cos 120^\circ & \sin 120^\circ \\ \sin 120^\circ & -\cos 120^\circ \end{pmatrix} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}.$$
2. **Matrix B:**
Enlargement scale factor 2:
$$\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$
3. **Matrix C:**
T_1 followed by T_2 means $\mathbf{C} = \mathbf{BA}$.
$$\begin{aligned} \mathbf{C} &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -0.5 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0.5 \end{pmatrix} \\ &= \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \end{aligned}$$
4. **Image of $Q(2, 0)$:**
$$\begin{aligned} \mathbf{q}' &= \mathbf{Cq} \\ &= \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} \end{aligned}$$
The image is $(-2, 2\sqrt{3})$.

5. **Finding R:**
We need to find $\mathbf{r}$ such that $\mathbf{Cr} = \mathbf{r}'$. We use the given inverse matrix $\mathbf{r} = \mathbf{C}^{-1}\mathbf{r}'$.

$$\begin{aligned}\mathbf{r} &= \begin{pmatrix} -0.25 & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & 0.25 \end{pmatrix} \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix} \\ &= \begin{pmatrix} -0.25(2) + \frac{\sqrt{3}}{4}(2\sqrt{3}) \\ \frac{\sqrt{3}}{4}(2) + 0.25(2\sqrt{3}) \end{pmatrix} \\ &= \begin{pmatrix} -0.5 + 1.5 \\ \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}\end{aligned}$$

So $R(1, \sqrt{3})$.

Ex 39: Consider two transformations:

- T_1: A shear parallel to the x-axis that maps $(1,0)$ to $(1,0)$ and $(0,1)$ to $(2,1)$.
- T_2: A rotation of $90°$ anti-clockwise about the origin.

1. Find the matrix $\mathbf{A}$ representing T_1.
2. Find the matrix $\mathbf{B}$ representing T_2.
3. Find the matrix $\mathbf{C}$ representing the composite transformation T_1 followed by T_2.
4. Find the image of the point $P(1,1)$ under this composite transformation.
5. Find the coordinates of point S such that its image under this composite transformation is $S'(-1,5)$.
(You are given that $\mathbf{C}^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$).

Answer:

1. **Matrix A:**
The columns of the matrix are the images of the basis vectors $\mathbf{i}(1,0)$ and $\mathbf{j}(0,1)$.
$T_1(\mathbf{i}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T_1(\mathbf{j}) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$
2. **Matrix B:**
Rotation of $90°$ anti-clockwise:
$$\mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
3. **Matrix C:**
T_1 followed by T_2 means $\mathbf{C} = \mathbf{BA}$.
$$\begin{aligned}\mathbf{C} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\end{aligned}$$
4. **Image of $P(1,1)$:**
$$\begin{aligned}\mathbf{p}' &= \mathbf{Cp} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 3 \end{pmatrix}\end{aligned}$$
The image is $(-1,3)$.
5. **Finding S:**
We need to find $\mathbf{s}$ such that $\mathbf{Cs} = \mathbf{s}'$. We use the given inverse matrix $\mathbf{s} = \mathbf{C}^{-1}\mathbf{s}'$.
$$\begin{aligned}\mathbf{s} &= \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 1 \end{pmatrix}\end{aligned}$$
So $S(3,1)$.

F AREA AND DETERMINANT

F.1 FINDING THE AREA OF THE IMAGE

Ex 40: A rectangle has area 10 square units. It is transformed by the matrix $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$.
Find the area of the rectangle image.

$$\text{Area}(S') = \boxed{100} \text{ square units}$$

Answer:

$$\begin{aligned}\det(\mathbf{A}) &= (3)(4) - (1)(2) \\ &= 12 - 2 \\ &= 10\end{aligned}$$

The area scale factor is $|\det(\mathbf{A})| = 10$.

$$\begin{aligned}\text{Area}(S') &= 10 \times \text{Area}(S) \\ &= 10 \times 10 \\ &= 100 \text{ square units}\end{aligned}$$

Ex 41: A triangle has an area of 5 square units. It is transformed by the matrix $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$.
Find the area of the image of the triangle.

$$\text{Area}(S') = \boxed{30} \text{ square units}$$

Answer:

$$\begin{aligned}\det(\mathbf{B}) &= (2)(3) - (0)(1) \\ &= 6 - 0 \\ &= 6\end{aligned}$$

The area scale factor is $|\det(\mathbf{B})| = |6| = 6$.

$$\begin{aligned}\text{Area}(S') &= 6 \times \text{Area}(S) \\ &= 6 \times 5 \\ &= 30 \text{ square units}\end{aligned}$$

Ex 42: A polygon has an area of 4 square units. It is transformed by the matrix $\mathbf{C} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.
Find the area of the image of the polygon.

$$\text{Area}(S') = \boxed{8} \text{ square units}$$

Answer:

$$\begin{aligned}\det(\mathbf{C}) &= (1)(4) - (3)(2) \\ &= 4 - 6 \\ &= -2\end{aligned}$$

The area scale factor is the absolute value of the determinant: $|\det(\mathbf{C})| = |-2| = 2$.

$$\begin{aligned}\text{Area}(S') &= 2 \times \text{Area}(S) \\ &= 2 \times 4 \\ &= 8 \text{ square units}\end{aligned}$$

F.2 FINDING UNKNOWNS USING AREA

Ex 43: Let T be a linear transformation defined by the matrix $\mathbf{M} = \begin{pmatrix} k & 2 \\ 1 & 4 \end{pmatrix}$, where k is a constant.

A triangle with area 10 cm^2 is transformed by T into an image triangle with area 180 cm^2.

1. Write down an expression for the determinant of $\mathbf{M}$ in terms of k.
2. Find the two possible values of k.
3. Given that the transformation preserves the orientation of the shape (i.e., the determinant is positive), find the value of k.

Answer:

1. **Determinant:**

$$\det(\mathbf{M}) = (k)(4) - (2)(1) = 4k - 2.$$

2. **Possible values of k:**
The area scale factor is $|\det(\mathbf{M})|$.

$$\begin{aligned}\text{Area}(S') &= |\det(\mathbf{M})| \times \text{Area}(S) \\ 180 &= |4k - 2| \times 10 \\ 18 &= |4k - 2|\end{aligned}$$

This leads to two cases:

$$\begin{aligned}4k - 2 &= 18 \quad\text{or}\quad & 4k - 2 &= -18 \\ 4k &= 20 & 4k &= -16 \\ k &= 5 & k &= -4\end{aligned}$$

The possible values are $k = 5$ and $k = -4$.

3. **Orientation preserved:**
Preserving orientation implies $\det(\mathbf{M}) > 0$.
If $k = 5$, $\det(\mathbf{M}) = 18 > 0$.
If $k = -4$, $\det(\mathbf{M}) = -18 < 0$.
Therefore, $k = 5$.

Ex 44: Let T be a linear transformation defined by the matrix $\mathbf{A} = \begin{pmatrix} 3 & x \\ 1 & 2 \end{pmatrix}$, where x is a constant.

A parallelogram with area 5 cm^2 is transformed by T into an image with area 30 cm^2.

1. Write down an expression for the determinant of $\mathbf{A}$ in terms of x.
2. Find the two possible values of x.
3. Given that the transformation reverses the orientation of the shape (i.e., the determinant is negative), find the value of x.

Answer:

1. **Determinant:**

$$\det(\mathbf{A}) = (3)(2) - (x)(1) = 6 - x.$$

2. **Possible values of x:**
The area scale factor is $|\det(\mathbf{A})|$.

$$\begin{aligned}\text{Area}(S') &= |\det(\mathbf{A})| \times \text{Area}(S) \\ 30 &= |6 - x| \times 5 \\ 6 &= |6 - x|\end{aligned}$$

This leads to two cases:

$$\begin{aligned}6 - x &= 6 \quad\text{or}\quad & -(6 - x) &= -6 \\ -x &= 0 & -x &= -12 \\ x &= 0 & x &= 12\end{aligned}$$

The possible values are $x = 0$ and $x = 12$.

3. **Orientation reversed:**
Reversing orientation implies $\det(\mathbf{A}) < 0$.
If $x = 0$, $\det(\mathbf{A}) = 6 - 0 = 6 > 0$ (Preserves).
If $x = 12$, $\det(\mathbf{A}) = 6 - 12 = -6 < 0$ (Reverses).
Therefore, $x = 12$.