AFFINE TRANSFORMATIONS

A LINEAR TRANSFORMATIONS

A.1 APPLYING LINEAR TRANSFORMATIONS

Ex 1: Find the image of the point $P(3,-2)$ under the transformation defined by the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$.

$$P'(\square, \square)$$

Ex 2: Find the image of the point $Q(4,1)$ under the transformation defined by the matrix $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$.

$$Q'(\square, \square)$$

Ex 3: Find the image of the point $R(-2,5)$ under the transformation defined by the matrix $\mathbf{M} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}$.

$$R'(\square, \square)$$

Ex 4: Find the image of the point $S(2,3)$ under the transformation defined by the matrix $\mathbf{C} = \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}$.

$$S'(\square, \square)$$

B STANDARD LINEAR TRANSFORMATIONS

B.1 FINDING THE MATRIX REPRESENTATION

Ex 5: Find the matrix representing a vertical stretch by a factor of 4.

$$\mathbf{S} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

Ex 6: Find the matrix representing a horizontal stretch by a factor of 3.

$$\mathbf{S} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

Ex 7: Find the matrix representing a reflection in the x-axis.

$$\mathbf{M} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

Ex 8: Find the matrix representing a reflection in the y-axis.

$$\mathbf{M} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

Ex 9: Find the matrix representing a reflection in the line $y = x$.

$$\mathbf{M} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

Ex 10: Find the matrix representing an enlargement (homothety) centered at the origin with a scale factor of 2.

$$\mathbf{H} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

Ex 11: Find the matrix representing a rotation of $\frac{\pi}{2}$ radians anti-clockwise about the origin.

$$\mathbf{R} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

MCQ 12: The transformation T is a reflection in the line $y = x\sqrt{3}$.
Find the matrix $\mathbf{T}$ that represents this transformation.

- ☐ $\mathbf{T} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
- ☐ $\mathbf{T} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$
- ☐ $\mathbf{T} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$
- ☐ $\mathbf{T} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

MCQ 13: The transformation T is a reflection in the line $y = -x\sqrt{3}$.
Find the matrix $\mathbf{T}$ that represents this transformation.

- ☐ $\mathbf{T} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$
- ☐ $\mathbf{T} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
- ☐ $\mathbf{T} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$
- ☐ $\mathbf{T} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

B.2 APPLYING STANDARD LINEAR TRANSFORMATIONS

Ex 14: Find the image of the point $A(2,5)$ under a reflection in the x-axis.

$$A'(\square, \square)$$

Ex 15: Find the image of the point $A(2,0)$ under a rotation of $\frac{\pi}{4}$ radians (45°) anti-clockwise about the origin.

$$A'(\square, \square)$$

Ex 16: Find the image of the point $A(3,4)$ under a reflection in the y-axis.

$A'(\square, \square)$

Ex 17: Find the image of the point $A(1, 2)$ under an enlargement centered at the origin with a scale factor of 3.

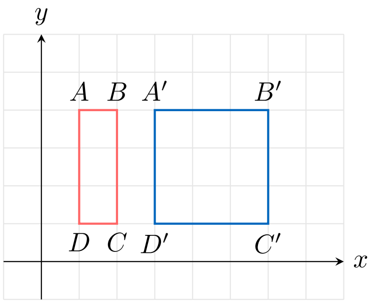
$A'(\square, \square)$

Ex 18: Find the image of the point $A(4, -1)$ under a rotation of π radians (180°) about the origin.

$A'(\square, \square)$

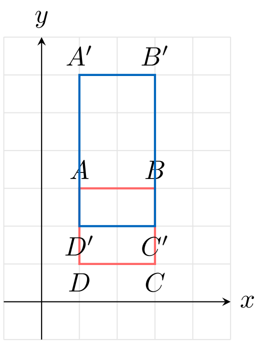
B.3 IDENTIFYING THE TRANSFORMATION

Ex 19: A linear transformation maps rectangle $ABCD$ onto square $A'B'C'D'$.



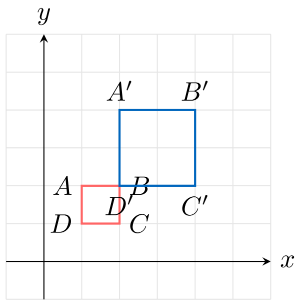
1. Identify the transformation.
2. Write down the transformation equation in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Ex 20: A linear transformation maps square $ABCD$ onto rectangle $A'B'C'D'$.



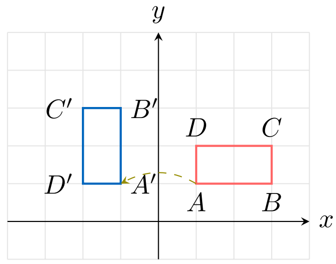
1. Identify the transformation.
2. Write down the transformation equation in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Ex 21: A linear transformation maps square $ABCD$ onto square $A'B'C'D'$.



1. Identify the transformation.
2. Write down the transformation equation in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Ex 22: A linear transformation maps rectangle $ABCD$ onto rectangle $A'B'C'D'$.



1. Identify the transformation.
2. Write down the transformation equation in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

C TRANSLATION

C.1 APPLYING TRANSLATIONS

Ex 23: Find the image of the point $A(1,2)$ under a translation by the vector $\mathbf{t} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

$$A'(\square, \square)$$

Ex 24: Find the image of the point $B(-2,3)$ under a translation by the vector $\mathbf{t} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

$$B'(\square, \square)$$

Ex 25: Find the image of the point $C(3,1)$ under a translation by the vector $\mathbf{t} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$.

$$C'(\square, \square)$$

D AFFINE TRANSFORMATION

D.1 APPLYING AN AFFINE TRANSFORMATION

Ex 26: Find the image of the point $P(2,1)$ under the affine transformation defined by $\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$$P'(\square, \square)$$

Ex 27: Find the image of the point $Q(1,3)$ under the affine transformation defined by $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$$Q'(\square, \square)$$

Ex 28: Find the image of the point $R(-1,2)$ under the affine transformation defined by $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -1 \\ -3 \end{pmatrix}$.

$$R'(\square, \square)$$

Ex 29: Find the image of the point $S(2,0)$ under the affine transformation defined by $\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

$$S'(\square, \square)$$

D.2 IDENTIFYING THE TRANSFORMATION

Ex 30: An affine transformation is defined by $\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -2 \\ 5 \end{pmatrix}$.

1. Find the image of the origin $O(0,0)$.
2. Find the image of the point $A(1,4)$.
3. Describe the transformation geometrically.

Ex 31: An affine transformation is defined by $\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

1. Find the image of the origin $O(0,0)$.
2. Find the image of the point $A(2,0)$.
3. Describe the transformation geometrically.

Ex 36: Let T_1 be a reflection in the x-axis and T_2 be a reflection in the y-axis.

1. Find the matrix $\mathbf{A}$ for T_1 and $\mathbf{B}$ for T_2.
2. Calculate the matrix for the composite transformation $T_2 \circ T_1$ (T_1 followed by T_2).
3. Identify the resulting transformation.

E COMPOSITION OF TRANSFORMATIONS

E.1 FINDING THE MATRIX REPRESENTATION

Ex 32: Find the single matrix representing a reflection in the y-axis followed by a horizontal stretch of scale factor 3.

$$\mathbf{T} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

Ex 33: Find the single matrix representing a rotation of 90° anti-clockwise followed by a vertical stretch of scale factor 2.

$$\mathbf{T} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

Ex 34: Find the single matrix representing a horizontal stretch of scale factor 4 followed by a reflection in the x-axis.

$$\mathbf{T} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

E.2 FINDING THE COMPOSITE MATRIX

Ex 35: Let T_1 be a rotation of 90° anti-clockwise and T_2 be a reflection in the line $y = x$.

1. Find the matrix $\mathbf{A}$ for T_1 and $\mathbf{B}$ for T_2.
2. Calculate the matrix for the composite transformation $T_2 \circ T_1$ (T_1 followed by T_2).
3. Identify the resulting transformation.

Ex 36: Let T_1 be a reflection in the x-axis and T_2 be a reflection in the y-axis.

1. Find the matrix $\mathbf{A}$ for T_1 and $\mathbf{B}$ for T_2.
2. Calculate the matrix for the composite transformation $T_2 \circ T_1$ (T_1 followed by T_2).
3. Identify the resulting transformation.

Ex 37: Let T_1 be a reflection in the line $y = x$ and T_2 be a reflection in the x-axis.

1. Find the matrix $\mathbf{A}$ for T_1 and $\mathbf{B}$ for T_2.
2. Calculate the matrix for the composite transformation $T_2 \circ T_1$ (T_1 followed by T_2).
3. Identify the resulting transformation.

E.3 CALCULATING AND INVERTING COMPOSITE MATRICES

Ex 38: Consider two transformations:

- T_1: A reflection in the line $y = x\sqrt{3}$.
- T_2: An enlargement centered at the origin with scale factor 2.

1. Find the matrix $\mathbf{A}$ representing T_1. (Recall: $\tan(60°) = \sqrt{3}$).
2. Find the matrix $\mathbf{B}$ representing T_2.

3. Find the matrix $\mathbf{C}$ representing the composite transformation T_1 followed by T_2.

4. Find the image of the point $Q(2,0)$ under this composite transformation.

5. Find the coordinates of point R such that its image under this composite transformation is $R'(2,2\sqrt{3})$.

   (You are given that $\mathbf{C}^{-1} = \begin{pmatrix} -0.25 & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & 0.25 \end{pmatrix}$).

Ex 39: Consider two transformations:

- T_1: A shear parallel to the x-axis that maps $(1,0)$ to $(1,0)$ and $(0,1)$ to $(2,1)$.
- T_2: A rotation of $90°$ anti-clockwise about the origin.

1. Find the matrix $\mathbf{A}$ representing T_1.
2. Find the matrix $\mathbf{B}$ representing T_2.
3. Find the matrix $\mathbf{C}$ representing the composite transformation T_1 followed by T_2.
4. Find the image of the point $P(1,1)$ under this composite transformation.
5. Find the coordinates of point S such that its image under this composite transformation is $S'(-1,5)$.

   (You are given that $\mathbf{C}^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$).

F AREA AND DETERMINANT

F.1 FINDING THE AREA OF THE IMAGE

Ex 40: A rectangle has area 10 square units. It is transformed by the matrix $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$.

Find the area of the rectangle image.

$$\text{Area}(S') = \boxed{} \text{ square units}$$

Ex 41: A triangle has an area of 5 square units. It is transformed by the matrix $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$.

Find the area of the image of the triangle.

$$\text{Area}(S') = \boxed{} \text{ square units}$$

Ex 42: A polygon has an area of 4 square units. It is transformed by the matrix $\mathbf{C} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

Find the area of the image of the polygon.

$$\text{Area}(S') = \boxed{} \text{ square units}$$

F.2 FINDING UNKNOWNS USING AREA

Ex 43: Let T be a linear transformation defined by the matrix $\mathbf{M} = \begin{pmatrix} k & 2 \\ 1 & 4 \end{pmatrix}$, where k is a constant.

A triangle with area 10 cm^2 is transformed by T into an image triangle with area 180 cm^2.

1. Write down an expression for the determinant of $\mathbf{M}$ in terms of k.
2. Find the two possible values of k.
3. Given that the transformation preserves the orientation of the shape (i.e., the determinant is positive), find the value of k.

Ex 44: Let T be a linear transformation defined by the matrix $\mathbf{A} = \begin{pmatrix} 3 & x \\ 1 & 2 \end{pmatrix}$, where x is a constant.

A parallelogram with area 5 cm^2 is transformed by T into an image with area 30 cm^2.

1. Write down an expression for the determinant of **A** in terms of x.
2. Find the two possible values of x.
3. Given that the transformation reverses the orientation of the shape (i.e., the determinant is negative), find the value of x.